

# Bôcher contractions of conformally superintegrable Laplace equations: Detailed computations

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## Abstract

These supplementary notes in the ArXiv are a companion to our paper “Bôcher contractions of conformally superintegrable Laplace equations”, <http://arxiv.org/abs/1512.09315>. They contain background material and the details of some of the extensive computations that couldn’t be put in the paper, due to space limitations.

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# 1 2D conformal superintegrability of the 2nd order

Systems of Laplace type are of the form

$$H\Psi \equiv \Delta_n\Psi + V\Psi = 0. \quad (1)$$

Here  $\Delta_n$  is the Laplace-Beltrami operator on a real or complex conformally flat  $nD$  Riemannian or pseudo-Riemannian manifold. We assume that all functions occurring in this paper are locally analytic, real or complex.) A conformal symmetry of this equation is a partial differential operator  $S$  in the variables  $\mathbf{x} = (x_1, \dots, x_n)$  such that  $[S, H] \equiv SH - HS = R_S H$  for some differential operator  $R_S$ . A conformal symmetry maps any solution  $\Psi$  of (1) to another solution. Two conformal symmetries  $S, S'$  are identified if  $S = S' + RH$  for some differential operator  $R$ , since they agree on the solution space of (1). (For short we will say that  $S = S'$ , mod  $(H)$  and that  $S$  is a symmetry if  $[S, H] = 0$ , mod  $(H)$ .) The system is *conformally superintegrable* for  $n > 2$  if there are  $2n - 1$  functionally independent conformal symmetries,  $S_1, \dots, S_{2n-1}$  with  $S_1 = H$ . It is second order conformally superintegrable if each symmetry  $S_i$  can be chosen to be a differential operator of at most second order.

For  $n = 2$  the definition must be restricted, since for a potential  $V = 0$  there will be an infinite dimensional space of conformal symmetries when  $n = 2$ ; every analytic function induces such symmetries.

**Comment 1** *Indeed necessary and sufficient conditions that  $S = u(x, y)\partial_x + v(x, y)\partial_y$  is a 1st order conformal symmetry for  $H = \Delta_2$  are that  $u$  and  $v$  satisfy the Cauchy-Riemann equations*

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v.$$

However, in this paper we are interested in multiparameter Laplace equations, i.e., those with potentials of the form  $V = \sum_{j=0}^s c_j V^{(j)}$  where the set  $\{V^{(j)}\}$  is linearly independent,  $V^{(0)} = 1$  and the  $c_j$  are arbitrary parameters. Thus we require that each symmetry be conformal for arbitrary choice of parameters  $c_j$  and, in particular for the special case  $V_0 = c_0$  where  $c_0$  is arbitrary. With this restriction we say that a 2D multiparameter Laplace equation is superintegrable if it admits 3 algebraically independent symmetries.

Every 2D Riemannian manifold is conformally flat, so we can always find a Cartesian-like coordinate system with coordinates  $\mathbf{x} = (x, y) \equiv (x_1, x_2)$  such that the Laplace equation takes the form

$$\tilde{H} = \frac{1}{\lambda(x, y)}(\partial_x^2 + \partial_y^2) + \tilde{V}(\mathbf{x}) = 0. \quad (2)$$

However, this equation is equivalent to the flat space equation

$$H \equiv \partial_x^2 + \partial_y^2 + V(\mathbf{x}) = 0, \quad V(\mathbf{x}) = \lambda(\mathbf{x})\tilde{V}(\mathbf{x}). \quad (3)$$

In particular, the conformal symmetries of (2) are identical with the conformal symmetries of (3). Indeed, denoting by  $\Lambda$  the operator of multiplication by the function  $\lambda(x, y)$  and using the operator identity  $[A, BC] = B[A, C] + [A, B]C$  we have

$$[S, H] = [S, \Lambda\tilde{H}] = \Lambda[S, \tilde{H}] + [S, \Lambda]\tilde{H} = \Lambda R\tilde{H} + [S, \Lambda]\tilde{H} = (\Lambda R\Lambda^{-1} + [S, \Lambda]\Lambda^{-1})H.$$

Thus without loss of generality we can assume the manifold is flat space with  $\lambda \equiv 1$ .

Since the Hamiltonians are formally self-adjoint, without loss of generality we can always assume that a 2nd order conformal symmetry  $S$  is formally self-adjoint and that a 1st order conformal symmetry  $L$  is skew-adjoint:

$$S = \frac{1}{\lambda} \sum_{k,j=1}^2 \partial_k \cdot (\lambda a^{kj}(\mathbf{x})) \partial_j + W(\mathbf{x}) \equiv S_0 + W, \quad a^{jk} = a^{kj} \quad (4)$$

$$L = \sum_{k=1}^2 \left( a^k(\mathbf{x}) \partial_k + \frac{\partial_k(\lambda a^k)}{2\lambda} \right). \quad (5)$$

$$[S, H] = (R^{(1)}(\mathbf{x}) \partial_x + (R^{(2)}(\mathbf{x}) \partial_y) H, \quad (6)$$

$$[L, H] = R(\mathbf{x}) H, \quad (7)$$

for some functions  $R^{(j)}(\mathbf{x}), R(\mathbf{x})$ .

Equating coefficients of the partial derivatives on both sides of (6), we obtain the conditions

$$a_i^{ii} = 2a_j^{ij} + a_i^{jj}, i \neq j \quad (8)$$

and

$$W_j = \sum_{s=1}^2 a^{sj} V_s + a_j^{jj} V, \quad k, j = 1, 2. \quad (9)$$

(Here a subscript  $j$  on  $a^{\ell m}$ ,  $V$  or  $W$  denotes differentiation with respect to  $x_j$ .) The requirement that  $\partial_x W_2 = \partial_y W_1$  leads from (9) to the second order (conformal) Bertrand-Darboux partial differential equations for the potential:

$$a^{12}(V_{11} - V_{22}) + (a^{22} - a^{11})V_{12} + (a_1^{12} + a_2^{22} - a_2^{11})V_1 + (a_1^{22} - a_1^{11} - a_2^{12})V_2 + 2A_{12}^{12}V = 0. \quad (10)$$

Furthermore, we can always add the trivial conformal symmetry  $\rho(\mathbf{x})H$  to  $S$ .

Equating coefficients of the partial derivatives on both sides of (7), we obtain the conditions

$$a_1^2 + a_2^1 = 0; \frac{R(\mathbf{x})}{2} = a_1^1 = a_2^2, \quad 2a_1^1 V + a^1 V_1 + a^2 V_2 = 0.. \quad (11)$$

In general the spaces of 1st and 2nd order symmetries could be infinite dimensional. However, the requirement that  $H$  have a multiparameter potential reduces the possible symmetries to a finite dimensional space. Indeed each such symmetry must necessarily be a symmetry for the potential  $V = c_0$  where  $c_0$  is an arbitrary parameter. Thus the conformal Bertrand-Darboux condition for a 2nd order symmetry yields the requirement  $\partial_{xy}(a^{11} - a^{22}) = 0$ . Furthermore we can always assume, say,  $a^{11} = 0$ . The result is that the pure derivative terms  $S_0$  belong to the space spanned by symmetrized products of the conformal Killing vectors

$$P_1 = \partial_x, \quad P_2 = \partial_y, \quad J = x_1 \partial_y - y \partial_x, \quad D = x \partial_x + y \partial_y, \quad (12)$$

$$K_1 = (x^2 - y^2) \partial_x + 2xy \partial_y, \quad K_2 = (y^2 - x^2) \partial_y + 2xy \partial_x.$$

and terms  $g(\mathbf{x})(\partial_x^2 + \partial_y^2)$  where  $g$  is an arbitrary function. For a given multiparameter potential only a subspace of these conformal tensors occurs. This is for two reasons. First the conformal Bertrand-Darboux equations restrict the allowed Killing tensors. Second, on the hypersurface  $\mathcal{H} = 0$  in phase space all symmetries  $g(\mathbf{x})\mathcal{H}$  vanish, so any two symmetries differing by  $g(\mathbf{x})\mathcal{H}$  can be identified.

Similarly the requirement that a 1st order conformal symmetry  $L$  be a symmetry for the potential  $V = c_0$  leads to the requirements  $a_x^1 = a_y^2 = R = 0$  so, in particular,  $L$  is a true (not just conformal) symmetry. Therefore its pure derivative part must be a linear combination of the Euclidean Killing vectors  $\partial_x$ ,  $\partial_y$ ,  $x\partial_y - y\partial_x$ .

The following results are easy modifications of results for 3D conformal superintegrable systems proved in [15]. We give them for completeness. For a conformal superintegrable system with 3 2nd order symmetries there will be 2 independent conformal Bertrand-Darboux equations (the equation for the symmetry  $H$  is trivial) and the assumption of algebraic independence means that we can solve these equations for  $V_{22}$  and  $V_{12}$ :

$$\begin{aligned} V_{22} &= V_{11} + A^{22}V_1 + B^{22}V_2 + C^{22}V, \\ V_{12} &= A^{12}V_1 + B^{12}V_2 + C^{12}V \end{aligned} \quad (13)$$

Here the  $A^{ij}, B^{ij}, C^{ij}$  are functions of  $\mathbf{x}$  that can be calculated explicitly. Indeed if  $\mathcal{S}_1 = \sum_{k,j=1}^2 \partial_k \cdot (\ell^{kj}(x, y))\partial_j + W^{(1)}(x, y)$ ,  $\mathcal{S}_2 = \sum_{k,j=1}^2 \partial_k \cdot (b^{kj}(x, y)\partial_j) + W^{(2)}(x, y)$ ,  $\mathcal{H}$ , is a basis for the symmetries then

$$A^{12} = \frac{D_{(2)}}{D}, \quad A^{22} = \frac{D_{(3)}}{D}, \quad B^{12} = -\frac{D_{(0)}}{D}, \quad B^{22} = -\frac{D_{(1)}}{D}, \quad (14)$$

$$C^{12} = -\frac{D_{(5)}}{D}, \quad C^{22} = -\frac{D_{(4)}}{D}, \quad (15)$$

$$D = \det \begin{pmatrix} \ell^{11} - \ell^{22} & \ell^{12} \\ b^{11} - b^{22} & b^{12} \end{pmatrix}, \quad D_{(0)} = \det \begin{pmatrix} 3\ell_2^{12} & -\ell^{12} \\ 3b_2^{12} & -b^{12} \end{pmatrix},$$

$$D_{(1)} = \det \begin{pmatrix} 3\ell_2^{12} & \ell^{11} - \ell^{22} \\ 3b_2^{12} & b^{11} - b^{22} \end{pmatrix}, \quad D_{(2)} = \det \begin{pmatrix} 3\ell_1^{12} & \ell^{12} \\ 3b_1^{12} & b^{12} \end{pmatrix},$$

$$D_{(3)} = \det \begin{pmatrix} 3\ell_1^{12} & \ell^{11} - \ell^{22} \\ 3b_1^{12} & b^{11} - b^{22} \end{pmatrix},$$

$$D_{(4)} = \det \begin{pmatrix} 2\ell_{12}^{12} & \ell^{11} - \ell^{22} \\ 2b_{12}^{12} & b^{11} - b^{22} \end{pmatrix}, \quad D_{(5)} = \det \begin{pmatrix} 2\ell_{12}^{12} & -\ell^{12} \\ 2b_{12}^{12} & -b^{12} \end{pmatrix}.$$

The functions  $A^{22}, B^{22}, A^{12}, B^{12}, C^{22}, C^{12}$  are defined independent of the choice of basis for the 2nd order symmetries.

## 1.1 The integrability conditions for the potential

To determine the integrability conditions for the system (13) we first introduce the dependent variables  $Z^{(0)} = V$ ,  $Z^{(1)} = V_1$ ,  $Z^{(2)} = V_2$ ,  $Z^{(3)} = V_{11}$ , the vector

$$\mathbf{z}^{\text{tr}} = (Z^{(0)}, Z^{(1)}, Z^{(2)}, Z^{(3)}), \quad (16)$$

and the matrices

$$\mathbf{A}^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ C^{12} & A^{12} & B^{12} & 0 \\ C^{13} & A^{13} & B^{13} & B^{12} - A^{22} \end{pmatrix}, \quad (17)$$

$$\mathbf{A}^{(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ C^{12} & A^{12} & B^{12} & 0 \\ C^{22} & A^{22} & B^{22} & 1 \\ C^{23} & A^{23} & B^{23} & A^{12} \end{pmatrix}, \quad (18)$$

where

$$\begin{aligned} A^{13} &= A_2^{12} - A_1^{22} + B^{12}A^{22} + A^{12}A^{12} - B^{22}A^{12} - C^{22}, \\ B^{13} &= B_2^{12} - B_1^{22} + A^{12}B^{12} + C^{12}, \\ C^{13} &= C_2^{12} - C_1^{22} + A^{12}C^{12} - B^{22}C^{12} + B^{12}C^{22}, \\ A^{23} &= A_1^{12} + B^{12}A^{12} + C^{12}, \quad B^{23} = B_1^{12} + B^{12}B^{12}, \\ C^{23} &= B^{12}C^{12} + C_1^{12}. \end{aligned} \quad (19)$$

Then the integrability conditions for the system

$$\partial_{x_j} \mathbf{z} = \mathbf{A}^{(j)} \mathbf{z} \quad j = 1, 2, \quad (20)$$

must hold. They are

$$\mathbf{A}_i^{(j)} - \mathbf{A}_j^{(i)} = \mathbf{A}^{(i)} \mathbf{A}^{(j)} - \mathbf{A}^{(j)} \mathbf{A}^{(i)} \equiv [\mathbf{A}^{(i)}, \mathbf{A}^{(j)}]. \quad (21)$$

Suppose the integrability conditions for system (13) are satisfied identically. In this case we say that the potential is *nondegenerate*. Otherwise the potential is *degenerate*. If  $V$  is nondegenerate then at any point  $\mathbf{x}_0$ , where the  $A^{ij}, B^{ij}, C^{ij}$  are defined and analytic, there is a unique solution  $V(\mathbf{x})$  with arbitrarily prescribed values of  $V(\mathbf{x}_0)$ ,  $V_1(\mathbf{x}_0)$ ,  $V_2(\mathbf{x}_0)$ ,  $V_{11}(\mathbf{x}_0)$ . The points  $\mathbf{x}_0$  are called *regular*. The points of singularity for the  $A^{ij}, B^{ij}, C^{ij}, D^{ij}$  form a manifold of dimension  $< 2$ . Degenerate potentials depend on fewer parameters. (For example, we could have that the integrability conditions are not satisfied identically. Or a first order conformal symmetry might exist and this would imply a linear condition on the first derivatives of  $V$  alone.)

Note that for a nondegenerate potential the solution space of (13) is exactly 4-dimensional, i.e. the potential depends on 4 parameters. Degenerate potentials depend on  $< 4$  parameters. Note also that the integrability conditions depend only on the free parts  $\ell^{jk}, b^{jk}$  of the conformal symmetry basis, not on the potential terms  $V, W^{(1)}, W^{(2)}$ . If the integrability conditions are satisfied identically, then the equations for the potential terms can be solved.

## 1.2 The conformal Stäckel transform

We quickly review the concept of the Stäckel transform [19] and extend it to conformally superintegrable systems. Suppose we have a second order *conformal* superintegrable system

$$H = \frac{1}{\lambda(x, y)}(\partial_{xx} + \partial_{yy}) + V(x, y) = 0, \quad H = H_0 + V. \quad (22)$$

and suppose  $U(x, y)$  is a particular solution of equations (13), nonzero in an open set. The *conformal Stäckel transform* of (22), induced by  $U$ , is the (Helmholtz) system

$$\tilde{H} = E, \quad \tilde{H} = \frac{1}{\tilde{\lambda}}(\partial_{xx} + \partial_{yy}) + \tilde{V}, \quad \tilde{\lambda} = \lambda U, \quad \tilde{V} = \frac{V}{U} \quad (23)$$

**Theorem 1** *The transformed (Helmholtz) system (23) is truly superintegrable.*

**Proof :** Let  $S = S_0 + W$  be a second order conformal symmetry of  $H$  and  $S_U = S_0 + W_U$  be the special case that is in conformal involution with  $\frac{1}{\lambda}(\partial_{xx} + \partial_{yy}) + U$ . Then

$$[S, H] = R_{S_0}H, \quad [S_U, H_0 + U] = R_{S_0}(H_0 + U), \quad [S_0, H_0] = R_{S_0}H_0$$

and  $\tilde{S} = S - \frac{W_U}{U}H$  is a corresponding true symmetry of  $\tilde{H}$ . Indeed,

$$\begin{aligned} [\tilde{S}, \tilde{H}] &= [S, U^{-1}H] - [\frac{W_U}{U}H, \frac{1}{U}H] = U^{-1}R_{S_0}H - U^{-1}[S_0, U]U^{-1}H \\ &\quad - U^{-1}[W_U, H_0]U^{-1}H = U^{-1}R_{S_0}H - U^{-1}R_{S_0}H = 0. \end{aligned}$$

This transformation of second order symmetries preserves linear and algebraic independence. Thus the transformed system is Helmholtz superintegrable.  $\square$

Note that if  $H\Psi = 0$  then  $\tilde{S}\Psi = S\Psi$  and  $H(S\Psi) = 0$  so  $S$  and  $\tilde{S}$  agree on the null space of  $H$  and they preserve this null space.

There is a similar result for first order conformal symmetries  $L$ .

**Corollary 1** *Let  $L$  be a first order conformal symmetry of the superintegrable system (22) and suppose  $U(\mathbf{x})$  is a particular solution of equations (13), nonzero in an open set. Then  $L$  is a true symmetry of the Helmholtz superintegrable system (23):  $[L, \tilde{H}] = 0$ .*

**Proof:** By assumption,  $[L, H] = R_L(\mathbf{x})H = R_L(H_0 + V)$  where  $R_L$  is a function. Thus,  $[L, H_0] = R_LH_0$ ,  $[L, V] = R_LV$ , so also  $[L, U] = R_LU$ . Then

$$\begin{aligned} [L, \tilde{H}] &= [L, U^{-1}H] = U^{-1}[L, H] - U^{-1}[L, U]U^{-1}H \\ &= U^{-1}RH - U^{-1}RUU^{-1}H = U^{-1}RH - U^{-1}RH = 0. \end{aligned}$$

$\square$

These results show that any second order conformal Laplace superintegrable system admitting a nonconstant potential  $U$  can be Stäckel transformed to a Helmholtz superintegrable system. This operation is invertible, but the inverse is not a Stäckel transform. By choosing all possible special potentials  $U$  associated with the fixed Laplace system (22) we generate the equivalence class of all Helmholtz superintegrable systems (23) obtainable through this process. As is easy to check, any two Helmholtz superintegrable systems lie in the same equivalence class if and only if they are Stäckel equivalent in the standard sense. All Helmholtz superintegrable systems are related to conformal Laplace systems in this way, so the study of all Helmholtz superintegrability on conformally flat manifolds can be reduced to the study of all conformal Laplace superintegrable systems on flat space.

**Theorem 2** *There is a one-to-one relationship between flat space conformally superintegrable Laplace systems with nondegenerate potential and Stäckel equivalence classes of superintegrable Helmholtz systems with nondegenerate potential on conformally flat spaces.*

Indeed, let

$$(H_1 - E_1)\Psi = 0, \quad (H_2 - E_2)\Psi = 0, \quad (24)$$

be Schrödinger eigenvalue equations where

$$H_j - E_j = \frac{1}{\lambda_j(x, y)}(\partial_{xx} + \partial_{yy} + V^{(j)}) - E_j, \quad j = 1, 2,$$



and

$$V = V^{(1)} + E_1 \lambda_1 = V^{(2)} + E_2 \lambda_2 \quad (25)$$

is a nondegenerate potential for the conformally superintegrable system

$$\partial_{xx} + \partial_{yy} + V = 0. \quad (26)$$

Suppose  $\{\lambda_1, \lambda_2\}$  is a linearly independent set (otherwise there is nothing to prove). Then we can find a potential basis for  $V$  of the form

$$\begin{aligned} V(x, y) &= -E_1 \lambda_1(x, y) - E_2 \lambda_2(x, y) + k_3 U^{(3)}(x, y) + k_4 U^{(4)}(x, y) \\ &= -E_1 \lambda_1 - E_2 \lambda_2 + \tilde{V} \end{aligned}$$

where  $\{\lambda_1, \lambda_2, U^{(3)}, U^{(4)}\}$  is a linearly independent set. Dividing (26) by  $\lambda_1, \lambda_2$ , respectively, we see that systems (24) are regular superintegrable with nondegenerate (3-parameter) potentials. Furthermore, multiplying the first system (24) by  $\lambda^{(1)}/\lambda^{(2)}$  we see that it is Stäckel equivalent to the second system. Conversely, if systems (24) are regular superintegrable and equality (25) holds, then it is easy to verify that system (26) is conformally superintegrable with nondegenerate (4-parameter) potential.

Even for true Helmholtz superintegrable systems there are good reasons to add a seemingly trivial constant to the potentials. Thus, for a Stäckel transform induced by the function  $U^{(1)}$ , we can take the original system to have Hamiltonian

$$H = H_0 + V = H_0 + U^{(1)} \alpha_1 + U^{(2)} \alpha_2 + U^{(3)} \alpha_3 + \alpha_4 \quad (27)$$

where  $\{U^{(1)}, U^{(2)}, U^{(3)}, 1\}$  is a basis for the 4-dimensional potential space. A 2nd order symmetry  $S$  would have the form

$$S = S_0 + W^{(1)} \alpha_1 + W^{(2)} \alpha_2 + W^{(3)} \alpha_3.$$

The Stäckel transformed Hamiltonian and symmetry take the form

$$\tilde{H} = \frac{1}{U^{(1)}} H_0 + \frac{U^{(1)} \alpha_1 + U^{(2)} \alpha_2 + U^{(3)} \alpha_3 + \alpha_4}{U^{(1)}}, \quad \tilde{S} = S - W^{(1)} \tilde{H}.$$

Note that the parameter  $\alpha_1$  cancels out of the expression for  $\tilde{S}$ ; it is replaced by  $-\alpha_4$ . Now suppose that  $\Psi$  is a formal eigenfunction of  $H$  (not required to be normalizable):  $H\Psi = E\Psi$ . If we choose the parameter  $\alpha_4 = -E$  in (27) then, in terms of this redefined  $H$ , we have  $H\Psi = 0$ . It follows immediately that  $\tilde{S}\Psi = S\Psi$ . Thus, for the 3-parameter system  $H'$  and the Stäckel transform  $\tilde{H}'$ ,

$$H' = H_0 + V' = H_0 + U^{(1)} \alpha_1 + U^{(2)} \alpha_2 + U^{(3)} \alpha_3,$$

$$\tilde{H}' = \frac{1}{U^{(1)}} H_0 + \frac{-U^{(1)} E + U^{(2)} \alpha_2 + U^{(3)} \alpha_3}{U^{(1)}},$$

we have  $H'\Psi = E\Psi$  and  $\tilde{H}'\Psi = -\alpha_1\Psi$ . It follows that The effect of the Stäckel transform is to replace  $\alpha_1$  by  $-E$  and  $E$  by  $-\alpha_1$ . Further, since  $S$  and  $\tilde{S}$  don't depend on the choice of  $\alpha_4$  we see that these operators must agree on eigenspaces of  $H'$

We know that the symmetry operators of all 2nd order nondegenerate superintegrable systems in 2D generate a quadratic algebra of the form

$$\begin{aligned} [R, S_1] &= f^{(1)}(S_1, S_2, \alpha_1, \alpha_2, \alpha_3, H'), \quad [R, S_2] = f^{(2)}(S_1, S_2, \alpha_1, \alpha_2, \alpha_3, H'), \\ R^2 &= f^{(3)}(S_1, S_2, \alpha_1, \alpha_2, \alpha_3, H'), \end{aligned} \quad (28)$$

where  $\{S_1, S_2, H\}$  is a basis for the 2nd order symmetries and  $\alpha_1, \alpha_2, \alpha_3$  are the parameters for the potential, [20, 21, 12]. It follows from the above considerations that the effect of a Stäckel transform generated by the potential function  $U^{(1)}$  is to determine a new superintegrable system with structure

$$[\tilde{R}, \tilde{S}_1] = f^{(1)}(\tilde{S}_1, \tilde{S}_2, -\tilde{H}', \alpha_2, \alpha_3, -\alpha_1), \quad (29)$$

$$[R, \tilde{S}_2] = f^{(2)}(\tilde{S}_1, \tilde{S}_2, -\tilde{H}', \alpha_2, \alpha_3, -\alpha_1),$$

$$R^2 = f^{(3)}(\tilde{S}_1, \tilde{S}_2, -\tilde{H}', \alpha_2, \alpha_3, -\alpha_1).$$

Of course, the switch of  $\alpha_1$  and  $H'$  is only for illustration; there is a Stäckel transform that replaces any  $\alpha_j$  by  $-H'$  and  $H'$  by  $-\alpha_j$ .

Formulas (28) and (29) are just instances of the quadratic algebras of the superintegrable systems belonging to the equivalence class of a single nondegenerate conformally superintegrable Hamiltonian

$$\hat{H} = \partial_{xx} + \partial_{yy} + \sum_{j=1}^4 \alpha_j V^{(j)}(x, y). \quad (30)$$

Let  $\hat{S}_1, \hat{S}_2, \hat{H}$  be a basis of 2nd order conformal symmetries of  $\hat{H}$ . From the above discussion we can conclude the following.

**Theorem 3** *The symmetries of the 2D nondegenerate conformal superintegrable Hamiltonian  $\hat{H}$  generate a quadratic algebra*

$$[\hat{R}, \hat{S}_1] = f^{(1)}(\hat{S}_1, \hat{S}_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4), \quad [\hat{R}, \hat{S}_2] = f^{(2)}(\hat{S}_1, \hat{S}_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4), \quad (31)$$

$$\hat{R}^2 = f^{(3)}(\hat{S}_1, \hat{S}_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4),$$

where  $\hat{R} = [\hat{S}_1, \hat{S}_2]$  and all identities hold mod  $(\hat{H})$ . A conformal Stäckel transform generated by the potential  $V^{(j)}(x, y)$  yields a nondegenerate Helmholtz superintegrable Hamiltonian  $\tilde{H}$  with quadratic algebra relations identical to (31), except that we make the replacements  $\hat{S}_\ell \rightarrow \tilde{S}_\ell$  for  $\ell = 1, 2$  and  $\alpha_j \rightarrow -\tilde{H}$ . These modified relations (30) are now true identities, not mod  $(\hat{H})$ .

Note that expressions (31) define a true quadratic algebra, interpreted mod  $(\hat{H})$ . They differ from the quadratic algebra for a Helmholtz system in that the Hamiltonian doesn't appear, whereas there is an extra parameter. The quadratic algebras of all Helmholtz systems obtained from  $\hat{H}$  via conformal Stäckel transforms follow by simple substitution.

**Comment 2** *Every 2nd order conformal symmetry is of the form  $S = S_0 + W$  where  $S_0$  is a 2nd order element of the enveloping algebra of  $so(4, \mathbb{C})$ . The dimension of this space of 2nd order elements is 21 but there is an 11-dimensional subspace of symmetries congruent to 0 mod  $H_0$  where  $H_0 = P_1^2 + P_2^2$ . A basis for this subspace is*

$$P_1^2 + P_2^2 \sim 0, \quad J^2 + D^2 \sim 0, \quad K_1^2 + K_2^2 \sim 0, \quad \{P_1, K_2\} + 2JD \sim 0,$$

$$\{P_1, J\} - \{P_2, D\} \sim 0, \quad \{P_1, K_1\} - \{P_2, K_2\} \sim 0, \quad \{J, K_1\} + \{D, K_2\} \sim 0,$$

$$\{P_1, D\} + \{P_2, J\} \sim 0, \quad \{P_1, K_2\} + \{P_2, K_1\} \sim 0, \quad \{J, K_2\} - \{D, K_1\} \sim 0,$$

$$4J^2 + \{P_1, K_1\} + \{P_2, K_2\} \sim 0.$$

Thus mod  $H_0$  the space of 2nd order symmetries is 10-dimensional.

### 1.3 Contractions of conformal superintegrable systems with potential induced by generalized Inönü-Wigner contractions

The basis symmetries  $\mathcal{S}^{(j)} = \mathcal{S}_0^{(j)} + W^{(j)}$ ,  $\mathcal{H} = \mathcal{H}_0 + V$  of a nondegenerate 2nd order conformally superintegrable system determine a conformal quadratic algebra (31), and if the parameters of the potential are set equal to 0, the free system  $\mathcal{S}_0^{(j)}, \mathcal{H}_0$ ,  $j = 1, 2$  also determines a conformal quadratic algebra without parameters, which we call a *free conformal quadratic algebra*. The elements of this free algebra belong to the enveloping algebra of  $so(4, \mathbb{C})$  with basis (12). Since the system is nondegenerate the integrability conditions for the potential are satisfied identically and the full quadratic algebra can be computed from the free algebra, modulo a choice of basis for the 4-dimensional potential space. Once we choose a basis for  $so(4, \mathbb{C})$ , its enveloping algebra is uniquely determined by the structure constants. Structure relations in the enveloping algebra are continuous functions of the structure constants, so a contraction of one  $so(4, \mathbb{C})$  to itself induces a contraction of the enveloping algebras. Then the free conformal quadratic algebra constructed in the enveloping algebra will contract to another free quadratic algebra. (In [5] essentially the same argument was given in more detail for Helmholtz superintegrable systems on constant curvature spaces.)

In this paper we consider a family of contractions of  $so(4, \mathbb{C})$  to itself that we call Bôcher contractions. All these contractions are implemented via coordinate transformations. Suppose we have a conformal nondegenerate superintegrable system with free generators  $\mathcal{H}_0, \mathcal{S}_0^{(1)}, \mathcal{S}_0^{(2)}$  that determines the conformal and free conformal quadratic algebras  $Q$  and  $Q^{(0)}$  and has structure functions  $A^{ij}(\mathbf{x})$ ,  $B^{ij}(\mathbf{x})$ ,  $C^{ij}(\mathbf{x})$  in Cartesian coordinates  $\mathbf{x} = (x_1, x_2)$ . Further, suppose this system contracts to another nondegenerate system  $\mathcal{H}'_0, \mathcal{S}'_0^{(1)}, \mathcal{S}'_0^{(2)}$  with conformal quadratic algebra  $Q'^{(0)}$ . We show here that this contraction induces a contraction of the associated nondegenerate superintegrable system  $\mathcal{H} = \mathcal{H}_0 + V$ ,  $\mathcal{S}^{(1)} = \mathcal{L}_0^{(1)} + W^{(1)}$ ,  $\mathcal{S}^{(2)} = \mathcal{S}_0^{(2)} + W^{(2)}$ ,  $Q$  to  $\mathcal{H}' = \mathcal{H}'_0 + V'$ ,  $\mathcal{S}'^{(1)} = \mathcal{S}'_0^{(1)} + W'^{(1)}$ ,  $\mathcal{S}'^{(2)} = \mathcal{S}'_0^{(2)} + W'^{(2)}$ ,  $Q'$ . The point is that in the contraction process the symmetries  $\mathcal{H}'_0(\epsilon)$ ,  $\mathcal{S}'_0^{(1)}(\epsilon)$ ,  $\mathcal{S}'_0^{(2)}(\epsilon)$  remain continuous functions of  $\epsilon$ , linearly independent as quadratic forms, and  $\lim_{\epsilon \rightarrow 0} \mathcal{H}'_0(\epsilon) = \mathcal{H}'_0$ ,  $\lim_{\epsilon \rightarrow 0} \mathcal{S}'_0^{(j)}(\epsilon) = \mathcal{S}'_0^{(j)}$ . Thus the associated functions  $A^{ij}(\epsilon)$ ,  $B^{ij}(\epsilon)$ ,  $C^{ij}(\epsilon)$  will also be continuous functions of  $\epsilon$  and  $\lim_{\epsilon \rightarrow 0} A^{ij}(\epsilon) = A'^{ij}$ ,  $\lim_{\epsilon \rightarrow 0} B^{ij}(\epsilon) = B'^{ij}$ ,  $\lim_{\epsilon \rightarrow 0} C^{ij}(\epsilon) = C'^{ij}$ . Similarly, the integrability conditions for the potential equations

$$\begin{aligned} V_{22}^{(\epsilon)} &= V_{11}^{(\epsilon)} + A^{22}(\epsilon)V_1^{(\epsilon)} + B^{22}(\epsilon)V_2^{(\epsilon)} + C^{22}(\epsilon)V^{(\epsilon)}, \\ V_{12}^{(\epsilon)} &= A^{12}(\epsilon)V_1^{(\epsilon)} + B^{12}(\epsilon)V_2^{(\epsilon)} + C^{12}(\epsilon)V^{(\epsilon)}, \end{aligned} \quad (32)$$

will hold for each  $\epsilon$  and in the limit. This means that the 4-dimensional solution space for the potentials  $V$  will deform continuously into the 4-dimensional solution space for the potentials  $V'$ . Thus the target space of solutions  $V'$  (and of the functions  $W'$ ) is uniquely determined by the free quadratic algebra contraction.

There is an apparent lack of uniqueness in this procedure, since for a nondegenerate superintegrable system one typically chooses a basis  $V^{(j)}$ ,  $j = 1, \dots, 4$  for the potential space and expresses a general potential as  $V = \sum_{j=1}^4 a_j V^{(j)}$ . Of course the choice of basis for the source

system is arbitrary, as is the choice for the target system. Thus the structure equations for the quadratic algebras and the dependence  $a_j(\epsilon)$  of the contraction constants on  $\epsilon$  will vary depending on these choices. However, all such possibilities are related by a basis change matrix.

## 2 Tetraspherical coordinates and relations with the 2-sphere and 2D flat space

The tetraspherical coordinates  $(x_1, \dots, x_4)$  satisfy  $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$ . They are projective coordinates on the null cone and have 3 degrees of freedom. Their principal advantage over flat space Cartesian coordinates is that the action of the conformal algebra (12) and of the conformal group  $\sim SO(4, \mathbb{C})$  is linearized in tetraspherical coordinates.

**Relation to Cartesian coordinates  $(x, y)$  and coordinates on the 2-sphere  $(s_1, s_2, s_3)$  :**

$$x_1 = 2XT, \quad x_2 = 2YT, \quad x_3 = X^2 + Y^2 - T^2, \quad x_4 = i(X^2 + Y^2 + T^2).$$

$$x = \frac{X}{T} = -\frac{x_1}{x_3 + ix_4}, \quad y = \frac{Y}{T} = -\frac{x_2}{x_3 + ix_4},$$

$$x = \frac{s_1}{1 + s_3}, \quad y = \frac{s_2}{1 + s_3},$$

$$s_1 = \frac{2x}{x^2 + y^2 + 1}, \quad s_2 = \frac{2y}{x^2 + y^2 + 1}, \quad s_3 = \frac{1 - x^2 - y^2}{x^2 + y^2 + 1},$$

$$H = \partial_{xx} + \partial_{yy} + \tilde{V} = (x_3 + ix_4)^2 \left( \sum_{k=1}^4 \partial_{x_k}^2 + V \right) = (1 + s_3)^2 \left( \sum_{j=1}^3 p_{s_j}^2 + V \right),$$

where  $\tilde{V} = (x_3 + ix_4)^2 V$  and

$$(1 + s_3) = -i \frac{(x_3 + ix_4)}{x_4}, \quad (1 + s_3)^2 = -\frac{(x_3 + ix_4)^2}{x_4^2},$$

$$s_1 = \frac{ix_1}{x_4}, \quad s_2 = \frac{ix_2}{x_4}, \quad s_3 = \frac{-ix_3}{x_4}.$$

Also,  $\sum_{k=1}^4 x_k \partial_{x_k} = 0$  and, classically,  $\sum_{k=1}^4 x_k p_k = 0$ .

**Relation to flat space and 2-sphere 1st order conformal constants of the motion:** We define

$$L_{jk} = x_j \partial_{x_k} - x_k \partial_{x_j}, \quad 1 \leq j, k \leq 4, \quad j \neq k,$$

where  $L_{jk} = -L_{kj}$ . The generators for flat space conformal symmetries are related to these via

$$P_1 = \partial_x = L_{13} + iL_{14}, \quad P_2 = \partial_y = L_{23} + iL_{24}, \quad D = iL_{34}, \quad (33)$$

$$J = L_{12}, \quad K_1 = L_{13} - iL_{14}, \quad K_2 = L_{23} - iL_{24}.$$

Here

$$D = x\partial_x + y\partial_y, \quad J = x\partial_y - y\partial_x, \quad K_1 = 2xD - (x^2 + y^2)\partial_x,$$

etc.

The generators for 2-sphere conformal constants of the motion are related to the  $L_{jk}$  via

$$L_{12} = J_{12} = s_1 \partial_{s_2} - s_2 \partial_{s_1}, \quad L_{13} = J_{13}, \quad L_{23} = J_{23},$$

$$L_{14} = -i\partial_{s_1}, \quad L_{24} = -i\partial_{s_2}, \quad L_{34} = -i\partial_{s_3}.$$

Note that in identifying tetraspherical coordinates we can always permute the parameters 1, 2, 3, 4. More generally, we can apply an arbitrary  $SO(4, \mathbb{C})$  transformation to the tetraspherical coordinates, so the above relations between Euclidean and tetraspherical coordinates are far from unique.

**2nd order conformal symmetries  $\sim H$ :** The 11-dimensional space of conformal symmetries  $\sim H$  has basis

$$\begin{aligned} & L_{12}^2 - L_{34}^2, \quad L_{13}^2 - L_{24}^2, \quad L_{23}^2 - L_{14}^2, \quad L_{12}^2 + L_{13}^2 + L_{23}^2, \\ & L_{12}L_{34} + L_{23}L_{14} - L_{13}L_{24}, \\ & \{L_{13}, L_{14}\} + \{L_{23}, L_{24}\}, \quad \{L_{13}, L_{23}\} + \{L_{14}, L_{24}\}, \quad \{L_{12}, L_{13}\} + \{L_{34}, L_{24}\}, \\ & \{L_{12}, L_{14}\} - \{L_{34}, L_{23}\}, \quad \{L_{12}, L_{23}\} - \{L_{34}, L_{14}\}, \quad \{L_{12}, L_{24}\} + \{L_{34}, L_{13}\}, \end{aligned} \quad (34)$$

All of this becomes much clearer if we make use of the decomposition  $so(4, \mathbb{C}) \equiv so(3, \mathbb{C}) \oplus so(3, \mathbb{C})$  and the functional realization of the Lie algebra. Setting

$$J_1 = \frac{1}{2}(L_{23} - L_{14}), \quad J_2 = \frac{1}{2}(L_{13} + L_{24}), \quad J_3 = \frac{1}{2}(L_{12} - L_{34}),$$

$$K_1 = \frac{1}{2}(L_{23} + L_{14}), \quad K_2 = \frac{1}{2}(L_{13} - L_{24}), \quad K_3 = \frac{1}{2}(L_{12} + L_{34}),$$

we have

$$[J_i, J_j] = \epsilon_{ijk} J_k, \quad [K_i, K_j] = \epsilon_{ijk} K_k, \quad [J_i, K_j] = 0.$$

In terms of the variable  $z = x + iy$ ,  $\bar{z} = x - iy$  we have

$$J_1 = \frac{1}{2}(i\partial_z - iz^2\partial_z), \quad J_2 = \frac{1}{2}(\partial_z + z^2\partial_z), \quad J_3 = iz\partial_z,$$

$$K_1 = \frac{1}{2}(-i\partial_{\bar{z}} + i\bar{z}^2\partial_{\bar{z}}), \quad K_2 = \frac{1}{2}(\partial_{\bar{z}} + \bar{z}^2\partial_{\bar{z}}), \quad K_3 = -i\bar{z}\partial_{\bar{z}},$$

so the  $J_i$  operators depend only on the variable  $z$  and the  $K_j$  operators depend only on the variable  $\bar{z}$ . Also

$$J_1^2 + J_2^2 + J_3^2 \equiv 0, \quad K_1^2 + K_2^2 + K_3^2 \equiv 0. \quad (35)$$

The space of 2nd order elements in the enveloping algebra is thus 21-dimensional and decomposes as  $A_z \oplus A_{\bar{z}} \oplus A_{z\bar{z}}$  where  $A_z$  is 5-dimensional with basis  $J_1^2, J_3^2, \{J_1, J_2\}, \{J_1, J_3\}, \{J_2, J_3\}$ ,  $A_{\bar{z}}$  is 5-dimensional with basis  $K_1^2, K_3^2, \{K_1, K_2\}, \{K_1, K_3\}, \{K_2, K_3\}$ , and  $A_{z\bar{z}}$  is 9-dimensional with basis  $J_i K_j$ ,  $1 \leq i, j \leq 3$ . Note that all of the elements of  $A_{z\bar{z}}$  are  $\sim H$ , whereas none of the nonzero elements of  $A_z, A_{\bar{z}}$  have this property. The 11 elements (34) include the relations (35). Here, the transposition  $J_i \leftrightarrow K_i$  is a conformal equivalence.

## 2.1 Classification of 2nd order conformally superintegrable systems with nondegenerate potential

With this simplification it becomes feasible to classify all conformally 2nd order superintegrable systems with nondegenerate potential. Since every such system has generators  $S^{(1)} = S_0^{(1)} + W_1(z, \bar{z})$ ,  $S^{(2)} = S_0^{(2)} + W_2(z, \bar{z})$ , it is sufficient to classify, up to  $SO(4, \mathbb{C})$  conjugacy, all free conformal

quadratic algebras with generators  $S_0^{(1)}, S_0^{(2)}, \text{ mod } H_0$ , and then to determine for which of these free conformal algebras the integrability conditions (21) hold identically, so that the system admits a nondegenerate potential  $\tilde{V}(z, \bar{z})$  which can be computed. The classification breaks up into the following possible cases:

- Case 1:  $S_0^{(1)}, S_0^{(2)} \in A_{\bar{z}}$ . (This is conformally equivalent to  $S_0^{(1)}, S_0^{(2)} \in A_{\bar{z}}$ .) The possible free conformal quadratic algebras of this type, classified up to  $SO(3, \mathbb{C})$  conjugacy mod  $J_1^2 + J_2^2 + J_3^2$  can easily be obtained from the computations in [5]. They are the pairs

1.

$$J_3^2, J_1^2$$

2.

$$J_3^2, \{J_1 + iJ_2, J_3\}$$

3.

$$J_3^2, \{J_1, J_3\}$$

4.

$$\{J_2, J_2 + iJ_1\}, \{J_2, J_3\}$$

5.

$$J_3^2, (J_1 + iJ_2)^2$$

6.

$$\{J_1 + iJ_2, J_3\}, (J_1 + iJ_2)^2. \quad (36)$$

Checking pairs 1) – 5) we find that they do not admit a nonzero potential, so they do not correspond to nondegenerate conformal superintegrable systems. This is in dramatic distinction to the results of [5] where for Helmholtz systems on constant curvature spaces there was a 1-1 relationship between free quadratic algebras and nondegenerate superintegrable systems. Pair 6), (36), does correspond to a superintegrable system, the singular case  $\tilde{V} = f(z)$  where  $f(z)$  is arbitrary. (This system is conformally Stäckel equivalent to the singular Euclidean system  $E_{15}$ .) Equivalently, the system in  $A_{\bar{z}}$  with analogous  $K$ -operators yields the potential  $\tilde{V} = f(\bar{z})$ , (65).

- Case 2:  $S_0^{(1)} = S_J^{(1)} + S_K^{(1)}, S_0^{(2)} = S_J^{(2)}$  where  $S_J^{(1)}, S_J^{(2)}$  are selected from one of the pairs 1) – 6) above and  $S_K^{(1)}$  is a nonzero element of  $A_{\bar{z}}$ . Again there is a conformally equivalent case where the roles of  $J_i$  and  $K_i$  are switched. To determine the possibilities for  $S_K^{(1)}$  we classify the 2nd order elements in the enveloping algebra of  $so(3, \mathbb{C})$  up to  $SO(3, \mathbb{C})$  conjugacy, mod  $K_1^2 + K_2^2 + K_3^2$ . From the computations in [5] we see easily that there are the following representatives for the equivalence classes:

a)

$$K_3^2$$

b)

$$K_1^2 + aK_2^2, a \neq 0, 1$$

c)

$$(K_1 + iK_2)^2$$

d)

$$K_3^2 + (K_1 + iK_2)^2$$

e)

$$\{K_3, K_1 + iK_2\}.$$

For pairs 1), 3), 4), 5) above and all choices  $a) - e)$  we find that the integrability conditions are never satisfied, so there are no corresponding nondegenerate superintegrable systems. For pair 2), however, we find that any choice  $a) - e)$  leads to the same nondegenerate superintegrable system [2, 2], (61). While it appears that there are multiple generators for this one system, each set of generators maps to any other set by a conformal Stäckel transformation and a change of variable. For pair 6), we find that any choice  $a) - e)$  leads to the same nondegenerate superintegrable system [4], (63). Again each set of generators maps to any other set by a conformal Stäckel transformation and a change of variable.

- Case 3:  $S_0^{(1)} = S_J^{(1)}$ ,  $S_0^{(2)} = S_J^{(2)} + S_K^{(2)}$  where  $S_J^{(1)}$ ,  $S_J^{(2)}$  are selected from one of the pairs 1) – 6) above and  $S_K^{(2)}$  is a nonzero element of  $A_{\bar{z}}$ . Again there is a conformally equivalent case where the roles of  $J_i$  and  $K_i$  are switched. To determine the possibilities for  $S_K^{(2)}$  we classify the 2nd order elements in the enveloping algebra of  $so(3, \mathbb{C})$  up to  $SO(3, \mathbb{C})$  conjugacy, mod  $K_1^2 + K_2^2 + K_3^2$ . They are  $a) - e)$  above. For pairs 1) – 4), 6) above and all choices  $a) - e)$  the integrability conditions are never satisfied, so there are no corresponding nondegenerate superintegrable systems. For pair 5), however, we find that any choice  $a) - e)$  leads to the same nondegenerate superintegrable system [2, 2], (61). Again each set of generators maps to any other set (and to any [2, 2] generators in Case 2) by a conformal Stäckel transformation and a change of variable.
- Case 4:  $S_0^{(1)} = S_J^{(1)}$ ,  $S_0^{(2)} = S_K^{(2)}$  where  $S_J^{(1)}$  is selected from one of the representatives  $a) - e)$  above and  $S_K^{(2)}$  is selected from one of the analogous representatives  $a) - e)$  expressed as  $K$ -operators. We find that each of the 25 sets of generators leads to the single conformally superintegrable system [0], (64), and each set of generators maps to any other set by a conformal Stäckel transformation and a change of variable.
- Case 5:  $S_0^{(1)} = S_J^{(1)} + S_K^{(1)}$ ,  $S_0^{(2)} = S_J^{(2)} + S_K^{(2)}$  where  $S_J^{(1)}$ ,  $S_J^{(2)}$  are selected from one of the pairs 1) – 6) above and  $S_K^{(1)}$ ,  $S_K^{(2)}$  are obtained from  $S_J^{(1)}$ ,  $S_J^{(2)}$ , respectively, by replacing each  $J_i$  by  $K_i$ . We find the following possibilities:
  - i)  $S_0^{(1)} = J_1^2 + K_1^2$ ,  $S_0^{(2)} = J_3^2 + K_3^2$ . This extends to the system [1, 1, 1, 1], (59).
  - ii)  $S_0^{(1)} = J_3^2 + K_3^2$ ,  $S_0^{(2)} = \{J_3, J_1 + iJ_2\} + \{K_3, K_1 + iK_2\}$ . This extends to the system [2, 1, 1], (60).
  - iii)  $S_0^{(1)} = J_3^2 + K_3^2$ ,  $S_0^{(2)} = \{J_1, J_3\} + \{K_1, K_3\}$ . This extends to the system [1, 1, 1, 1], (59) again, equivalent to the generators *i)* by a conformal Stäckel transformation and a change of variable.
  - iv)  $S_0^{(1)} = \{J_1, J_2 + iJ_1\} + \{K_1, K_2 + iK_1\}$ ,  $S_0^{(2)} = \{J_2, J_3\} + \{K_2, K_3\}$ . This does not extend to a conformal superintegrable system.
  - v)  $S_0^{(1)} = (J_1 + iJ_2)^2 + (K_1 + iK_2)^2$ ,  $S_0^{(2)} = J_3^2 + K_3^2$ . This extends to the system [2, 1, 1], (60) again, equivalent to the generators *ii)* by a conformal Stäckel transformation and a change of variable.
  - vi)  $S_0^{(1)} = \{J_3, J_1 + iJ_2\} + \{K_3, K_1 + iK_2\}$ ,  $S_0^{(2)} = (J_1 + iJ_2)^2 + (K_1 + iK_2)^2$ , which extends to the system [3, 1], (62).

This completes the classification.

**Example 1** We describe how apparantly distinct superintegrable systems of a fixed type are actually the same. In Case 2 consider the system with generators  $\{J_1 + iJ_2, J_3\} + (K_1 + iK_2)^2, (J_1 + iJ_2)^2$ . This extends to the conformally superintegrable system [4] with flat space Hamiltonian operator  $H_1 = \partial_{z\bar{z}} + V^{(1)}$  where

$$V^{(1)} = 2k_3 z\bar{z} + 2k_4 z + k_3 \bar{z}^3 + 3k_4 \bar{z}^2 + k_1 \bar{z} + k_2.$$

The system with generators  $\{J_1 + iJ_2, J_3\} + K_3^2 + (K_1 + iK_2)^2, (J_1 + iJ_2)^2$  again extends to the conformally superintegrable system [4]. Indeed, replacing  $z, \bar{z}$  by  $Z, \bar{Z}$  to distinguish the two systems, we find the 2nd flat space Hamiltonian operator  $H_2 = \partial_{Z\bar{Z}} + V^{(2)}$  where

$$V^{(2)} = \frac{c_3 \operatorname{arcsinh}^3(\bar{Z}) + 3c_4 \operatorname{arcsinh}^2(\bar{Z}) + (2c_3 Z + c_1) \operatorname{arcsinh}(\bar{Z}) + 2c_4 Z + c_2}{\sqrt{1 - \bar{Z}^2}}.$$

Now we perform a conformal Stäckel transform on  $H_2$  to obtain the new flat space system

$$\begin{aligned} \tilde{H}_2 = & \sqrt{1 - \bar{Z}^2} \partial_{Z\bar{Z}} + c_3 \operatorname{arcsinh}^3(\bar{Z}) + 3c_4 \operatorname{arcsinh}^2(\bar{Z}) \\ & + (2c_3 Z + c_1) \operatorname{arcsinh}(\bar{Z}) + 2c_4 Z + c_2. \end{aligned}$$

Making the change of variable  $\bar{Z} = \sinh W$ , we find

$$\tilde{H}_2 = \partial_{ZW} + c_3 W^3 + 3c_4 W^2 + (2c_3 Z + c_1)W + 2c_4 Z + c_2.$$

Thus, with the identifications  $Z = z$ ,  $W = \bar{z}$ ,  $c_i = k_i$ , we see that  $H_1 \equiv \tilde{H}_2$ .

## 2.2 Relation to separation of variables

Bôcher's analysis [13] involves symbols of the form  $[n_1, n_2, \dots, n_p]$  where  $n_1 + \dots + n_p = 4$ . These symbols are used to define coordinate surfaces as follows. Consider the quadratic forms

$$\Omega = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, \quad \Phi = \frac{x_1^2}{\lambda - e_1} + \frac{x_2^2}{\lambda - e_2} + \frac{x_3^2}{\lambda - e_3} + \frac{x_4^2}{\lambda - e_4} = 0. \quad (37)$$

If  $e_1, e_2, e_3, e_4$  are pairwise distinct, the elementary divisors of these two forms are denoted by the symbol  $[1, 1, 1, 1]$ . Given a point in 2D flat space with Cartesian coordinates  $(x^0, y^0)$ , there corresponds a set of tetraspherical coordinate  $(x_1^0, x_2^0, x_3^0, x_4^0)$ , unique up to multiplication by a nonzero constant. If we substitute these coordinates into expressions (37) we can verify that there are exactly 2 roots  $\lambda = \rho, \mu$  such that  $\Phi = 0$ . These are elliptic coordinates. It can be verified that they are orthogonal with respect to the metric  $ds^2 = dx^2 + dy^2$  and that they are  $R$ -separable for the Laplace equations  $(\partial_x^2 + \partial_y^2)\Theta = 0$  or  $(\sum_{j=1}^4 \partial_{x_j}^2)\Theta = 0$ . Now consider the potential

$$V_{[1,1,1,1]} = \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2} + \frac{a_4}{x_4^2}.$$

It turns out to be the only possible potential  $V$  such that the Laplace equation  $(\sum_{j=1}^4 \partial_{x_j}^2 + V)\Theta = 0$  is  $R$ -separable in elliptic coordinates for all choices of the parameters  $e_j$ . The separation is characterized by 2nd order conformal symmetry operators that are linear in the parameters



$e_j$ . In particular the symmetries span a 3-dimensional subspace of symmetries, so the system  $(\sum_{j=1}^4 \partial_{x_j}^2 + V_{[1,1,1,1]})\Theta = 0$  must be conformally superintegrable. We can write this as

$$H = (x_3 + ix_4)^2 (\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2 + \partial_{x_4}^2 + \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2} + \frac{a_4}{x_4^2}),$$

or in terms of flat space coordinates  $x, y$  as

$$H = \partial_x^2 + \partial_y^2 + \frac{a_1}{x^2} + \frac{a_2}{y^2} + \frac{4a_3}{(x^2 + y^2 - 1)^2} - \frac{4a_4}{(x^2 + y^2 + 1)^2}.$$

For the coordinates  $s_i, i = 1, 2, 3$  we obtain

$$H = (1 + s_3)^2 (\partial_{s_1}^2 + \partial_{s_2}^2 + \partial_{s_3}^2 - \frac{a_1}{s_1^2} - \frac{a_2}{s_2^2} - \frac{a_3}{s_3^2} - a_4).$$

The coordinate curves are described by  $[1, 1, 1, \infty]$  (because we can always transform to equivalent coordinates for which  $e_4 = \infty$ ) and the corresponding  $H\Theta = 0$  system is proportional to  $S_9$ , the eigenvalue equation for the generic potential on the 2-sphere, which separates variables in elliptic coordinates  $s_i^2 = \frac{(\rho - e_i)(\mu - e_i)}{(e_i - e_j)(e_i - e_k)}$  where  $(e_i - e_j)(e_i - e_k) \neq 0$  and  $i, j, k = 1, 2, 3$ . The quantum Hamiltonian when written using these coordinates is equivalent to

$$\mathcal{H} = \frac{1}{\rho - \mu} [P_\rho^2 - P_\mu^2] - \sum_{i=1}^3 a_i \frac{(e_i - e_j)(e_i - e_k)}{(\rho - e_i)(\mu - e_i)},$$

where  $P_\lambda = \sqrt{\prod_{i=1}^3 (\lambda - e_i)} \partial_\lambda$ .

### 3 Bôcher contractions

These are contractions of  $so(4, \mathbb{C})$  to itself that are induced by coordinate transformations on the null cone that Bôcher used to derive the separable coordinate systems for the flat space Laplace and wave equations, [13, 14]. In the following notes we shall usually list 6 symmetries for each superintegrable system  $[1, 1, 1, 1] - [4]$ , which is strictly the case for the analogous systems on the 2-sphere. However, these systems are defined on the null cone, which implies extra constraints, Therefore instead of 6 linearly independent symmetries we have only 3.

We start with the potential

$$V_{[1,1,1,1]} = \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2} + \frac{a_4}{x_4^2}, \quad (38)$$

and the system  $[1, 1, 1, 1]$  and use successive Bôcher contractions to derive the systems  $[2, 1, 1], [2, 2], [3, 1], [4]$  and  $[0]$ .

#### 3.1 The $[1, 1, 1, 1]$ to $[2, 1, 1]$ contraction

If two of the  $e_i$  in eqns (37) become equal, Bôcher shows that the process of making  $e_1 \rightarrow e_2$  together with suitable transformations of the  $a_i$ 's produces a conformally equivalent  $H$ . This corresponds to the choice of coordinate curves obtained by the Bôcher limiting process  $[1, 1, 1, 1] \rightarrow [2, 1, 1]$ , i.e.,

$$e_1 = e_2 + \epsilon^2, \quad x_1 \rightarrow \frac{iy_1}{\epsilon}, \quad x_2 \rightarrow \frac{y_1}{\epsilon} + \epsilon y_2, \quad x_j \rightarrow y_j, \quad j = 3, 4,$$

which results in the pair of quadratic forms

$$\Omega = 2y_1y_2 + y_3^2 + y_4^2 = 0, \quad \Phi = \frac{y_1^2}{(\lambda - e_2)^2} + \frac{2y_1y_2}{(\lambda - e_2)} + \frac{y_3^2}{(\lambda - e_3)} + \frac{y_4^2}{(\lambda - e_4)} = 0.$$

The coordinate curves with  $e_4 = \infty$  correspond to cyclides with elementary divisors  $[2, 1, 1]$ , [3], i.e.,  $\Phi = \frac{y_1^2}{(\lambda - e_2)^2} + \frac{2y_1y_2}{(\lambda - e_2)} + \frac{y_3^2}{(\lambda - e_3)} = 0$ .

**Comment 3** *Indeed, making the substitution  $\lambda = \frac{\alpha\lambda' + \beta}{\gamma\lambda' + \delta}$ ,  $e_i = \frac{\alpha e'_i + \beta}{\gamma e'_i + \delta}$  we do not change the family of surfaces described (see [13], page 59). In particular the second quadratic form becomes*

$$\Phi = \frac{y_1^2(\gamma e'_1 + \delta)^2}{(\lambda' - e'_1)^2(\alpha\delta - \beta\gamma)} + \frac{2y_1y_2}{\lambda' - e'_1} + \frac{y_3^2}{\lambda' - e'_3} + \frac{y_4^2}{\lambda' - e'_4} = 0.$$

Now if we let  $e'_1 = \infty$  we obtain essentially  $\Phi = \frac{y_1^2\gamma^2}{(\alpha\delta - \beta\gamma)} + \frac{y_3^2}{\lambda' - e'_3} + \frac{y_4^2}{\lambda' - e'_4} = 0$ , which means that we have degenerate elliptic coordinates of type 1 in the plane with coordinate curves denoted by  $[\infty, 1, 1]$ . If we took  $e'_4 = \infty$  we would obtain the coordinate curves of degenerate elliptic coordinates on the sphere with coordinate curves denoted by  $[2, 1, 1]$ . If we take  $e'_4 = \infty$  in generic tetracyclic coordinates we obtain elliptic coordinates on the 3-sphere with coordinate curves denoted by  $[1, 1, 1, \infty]$ . Our subsequent studies elaborate on these observations.

Note that the composite linear coordinate mapping

$$x_1 + ix_2 = \frac{i\sqrt{2}}{\epsilon}(x'_1 + ix'_2) + \frac{i\epsilon}{\sqrt{2}}(x'_1 - ix'_2), \quad x_1 - ix_2 = -\frac{i\epsilon}{\sqrt{2}}(x'_1 - ix'_2),$$

$$x_3 = x'_3, \quad x_4 = x'_4,$$

satisfies  $\lim_{\epsilon \rightarrow 0} \sum_{j=1}^4 x_j^2 = \sum_{j=1}^4 x'^2_j = 0$ , and induces a contraction of the Lie algebra  $so(4, \mathbb{C})$  to itself. An explicit computation yields

$$L'_{12} = L_{12}, \quad L'_{13} = -\frac{i}{\sqrt{2}\epsilon}(L_{13} - iL_{23}) - \frac{i\epsilon}{\sqrt{2}}L_{13}, \quad L'_{23} = -\frac{i}{\sqrt{2}\epsilon}(L_{13} - iL_{23}) - \frac{\epsilon}{\sqrt{2}}L_{13}$$

$$L'_{34} = L_{34}, \quad L'_{14} = -\frac{i}{\sqrt{2}\epsilon}(L_{14} - iL_{24}) - \frac{i\epsilon}{\sqrt{2}}L_{14}, \quad L'_{24} = -\frac{i}{\sqrt{2}\epsilon}(L_{14} - iL_{24}) - \frac{\epsilon}{\sqrt{2}}L_{14}.$$

This is the Bôcher contraction  $[1, 1, 1, 1] \rightarrow [2, 1, 1]$ .

### 3.1.1 Conformal Stäckel transforms of the $[1, 1, 1, 1]$ system

We write the parameters  $a_j$  defining the potential  $V_{[1, 1, 1, 1]}$  as a vector:  $(a_1, a_2, a_3, a_4)$ .

1. The potentials  $(1, 0, 0, 0)$ , and any permutation of the indices  $a_j$  generate conformal Stäckel transforms to  $S9$ .
2. The potentials  $(1, 1, 0, 0)$  and  $(0, 0, 1, 1)$  generate conformal Stäckel transforms to  $S7$ .
3. The potentials  $(1, 1, 1, 1)$ ,  $(0, 1, 0, 1)$ ,  $(1, 0, 1, 0)$ ,  $(0, 1, 1, 0)$  and  $(1, 0, 0, 1)$  generate conformal Stäckel transforms to  $S8$ .
4. The potentials  $(a_1, a_2, 0, 0)$ ,  $a_1a_2 \neq 0, a_1 \neq a_2$ , and any permutation of the indices  $a_j$ . generate conformal Stäckel transforms to  $D4B$ .
5. The potentials  $(1, 1, a, a)$ ,  $a \neq 0, 1$ , and any permutation of the indices  $a_j$ . generate conformal Stäckel transforms to  $D4C$ .

6. Each potential not proportional to one of these must generate a conformal Stäckel transform to a superintegrable system on a Koenigs space in the family  $K[1, 1, 1, 1]$ .

Now under the contraction  $[1, 1, 1, 1] \rightarrow [2, 1, 1]$  we have

$$V_{[1,1,1,1]} \xrightarrow{\epsilon \rightarrow 0} V_{[2,1,1]}$$

where

$$V_{[2,1,1]} = \frac{b_1}{(x'_1 + ix'_2)^2} + \frac{b_2(x'_1 - ix'_2)}{(x'_1 + ix'_2)^3} + \frac{b_3}{x_3'^2} + \frac{b_4}{x_4'^2}, \quad (39)$$

and

$$a_1 = -\frac{1}{2}\left(\frac{b_1}{\epsilon^2} + \frac{b_2}{2\epsilon^4}\right), \quad a_2 = -\frac{b_2}{4\epsilon^4}, \quad a_3 = b_3, \quad a_4 = b_4.$$

**Comment 4** We established the potential limit by direct computation. However this contraction can be understood in terms of the generic elliptic coordinates. We put  $e_1 = 0, e_2 = \epsilon^2$  and  $e_3 = e_3$  in the coordinates

$$s_1^2 = \frac{(x_1 - e_1)(x_2 - e_1)}{(e_1 - e_2)(e_1 - e_3)}, \quad s_2^2 = \frac{(x_1 - e_2)(x_2 - e_2)}{(e_2 - e_1)(e_2 - e_3)}, \quad s_3^2 = \frac{(x_1 - e_3)(x_2 - e_3)}{(e_3 - e_2)(e_3 - e_1)}.$$

We take as the generic potential  $V = \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}$  and subject the coefficients  $a_i$  to the contraction transformations

$$a_1 \rightarrow -\frac{b_1}{\epsilon^2} + \frac{b_2}{\epsilon^4}, \quad a_2 \rightarrow \frac{b_2}{\epsilon^4},$$

Under  $\epsilon \rightarrow 0$  we obtain

$$V \rightarrow \frac{b_1 e_3}{x_1 x_2} + b_2 \left[ \frac{-e_3(x_1 + x_2) + x_1 x_2}{x_1^2 x_2^2} \right] + \frac{e_3^2 a_3}{(x_1 - e_3)(x_2 - e_3)}.$$

This form of the potential is separable in type  $[2, 1, 1]$  coordinates, as can be seen from the relations

$$\frac{(x_1 - x_2)}{x_1 x_2} = \frac{1}{x_2} - \frac{1}{x_1}, \quad \frac{(x_1^2 - x_2^2)}{x_1^2 x_2^2} = \frac{1}{x_2^2} - \frac{1}{x_1^2},$$

$$\frac{(x_1 - x_2)}{(x_1 - e_3)(x_2 - e_3)} = \frac{1}{x_2 - e_3} - \frac{1}{x_1 - e_3}.$$

Consider another case, viz  $e_1 = 0, e_2 = \epsilon$  and  $e_3 = A\epsilon$ . From the transformations

$$a_1 \rightarrow \frac{c_1}{\epsilon^4} + \frac{c_2}{\epsilon^6} + \frac{b_3}{\epsilon^8}, \quad a_2 \rightarrow \frac{c_2}{\epsilon^6(A-1)} + \frac{c_3 A^2}{\epsilon^8(A-1)^2}, \quad a_3 \rightarrow \frac{c_3}{\epsilon^8(A-1)^2},$$

we obtain the limit of  $V$  as

$$V \rightarrow \frac{Ac_1}{x_1 x_2} - \frac{Ac_2(x_1 + x_2)}{x_1^2 x_2^2} + \frac{A^2 c_3(x_1^2 + x_1 x_2 + x_2^2)}{x_1^3 x_2^3}.$$

What we deduce from these examples is that if  $e_1$  is a root of the polynomial of the corresponding degenerate elliptic system on the complex sphere and it has multiplicity  $p$ , there are terms in the potential of the form

$$\frac{1}{(x_1 - x_2)} \left[ \frac{1}{(x_1 - e_1)^s} - \frac{1}{(x_2 - e_1)^s} \right], \quad s = 1 \cdots p$$

and the transformation of  $a_i$  can be determined.

**Examples 1** Using Cartesian coordinates  $x, y$ , we consider the Hamiltonian

$$H = \partial_x^2 + \partial_y^2 + \frac{a_1}{x^2} + \frac{a_2}{y^2} + \frac{4a_3}{(x^2 + y^2 - 1)^2} + \frac{4a_4}{(x^2 + y^2 + 1)^2}.$$

Multiplying on the left by  $x^2$  we obtain

$$\hat{H} = x^2(\partial_x^2 + \partial_y^2) + a_1 + a_2 \frac{x^2}{y^2} + 4a_3 \frac{x^2}{(x^2 + y^2 - 1)^2} - 4a_4 \frac{x^2}{(x^2 + y^2 + 1)^2},$$

the case  $\mathbf{a} = (1, 0, 0, 0)$ . This becomes more transparent if we introduce variables  $x = e^{-a}, y = r$ . The Hamiltonian  $\hat{H}$  can be written

$$\hat{H} = \partial_a^2 + \partial_r^2 + e^{-2a} \partial_r^2 + a_1 + a_2 \frac{e^{-2a}}{r^2} + a_3 \frac{4}{(e^{-a} + e^a(r^2 - 1))^2} - a_4 \frac{4}{(e^{-a} + e^a(r^2 + 1))^2}.$$

Recalling horospherical coordinates on the complex two sphere, viz.

$$s_1 = \frac{i}{2}(e^{-a} + (r^2 + 1)e^a), \quad s_2 = re^a, \quad s_3 = \frac{1}{2}(e^{-a} + (r^2 - 1)e^a)$$

we see that the Hamiltonian  $\hat{H}$  can be written as

$$\hat{H} = \partial_{s_1}^2 + \partial_{s_2}^2 + \partial_{s_3}^2 + a_1 + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2} + \frac{a_4}{s_1^2},$$

and this is explicitly the superintegrable system  $S_9$ .

Now consider the case  $\mathbf{a} = (0, 1, 0, 1)$  which for  $x = e^a \sin \varphi$ ,  $y = e^a \cos \varphi$  and conformal Stäckel multiplier

$$\left(\frac{1}{y^2} - \frac{4}{(x^2 + y^2 + 1)^2}\right) = e^{-2a} \left(\frac{1}{\cos^2 \varphi} - \frac{1}{\cosh^2 a}\right)$$

yields the Hamiltonian

$$\frac{1}{\left(\frac{1}{\cos^2 \varphi} - \frac{1}{\cosh^2 a}\right)} \left[ \partial_a^2 + \partial_\varphi^2 + \frac{a_1}{\sin^2 \varphi} + \frac{a_2 + a_4}{2} \left( \frac{1}{\cos^2 \varphi} + \frac{1}{\cosh^2 a} \right) + \frac{a_3}{\sinh^2 a} \right] + \frac{a_2 - a_4}{2},$$

which is just  $S_8$  in elliptic coordinates of type 1, the coordinates on the 2-sphere being taken as

$$s_1 + is_2 = \frac{1}{\cos \varphi \cosh a}, \quad s_1 - is_2 = \frac{\cos \varphi}{\cosh a} + \frac{\cosh a}{\cos \varphi} - \frac{1}{\cos \varphi \cosh a}, \quad s_3 = i \tan \varphi \tanh a,$$

where  $s_1^2 + s_2^2 + s_3^2 = 1$ .

Now consider the case  $\mathbf{a} = (1, 1, 0, 0)$ , with

$$x = e^{ia/2} \cos b, \quad y = e^{ia/2} \sin b.$$

If instead we use the variable  $B$  where  $\sin 2b = \frac{1}{\cosh B}$  then the Hamiltonian can be written

$$\partial_B^2 + \tanh B \partial_B - \frac{1}{\cosh^2 B} \partial_a^2 + b_1 \tanh B + b_2 \frac{1}{\sinh^2 a \cosh^2 B} + b_3 \frac{1}{\cosh^2 a \cosh^2 B} + b_0$$

which is directly Stäckel equivalent to  $S_7$ . A suitable choice of coordinates on the complex 2-sphere is

$$s_1 = \cosh a \cosh B, \quad s_2 = i \cosh a \sinh B, \quad s_3 = i \sinh a.$$

For the case  $\mathbf{a} = (b_1, b_2, 0, 0)$  the Stäckel multiplier (potential that induces the Stäckel transform) is  $b_1/x^2 + b_2/y^2$ . In terms of the coordinates  $x = e^v \cos \theta, y = e^v \sin \theta$  the Hamiltonian takes the form

$$H = \frac{\sin^2 2\theta}{2[(b_2 - b_1) \cos 2\theta + (b_1 - b_2)]} \left[ \partial_\theta^2 + \partial_v^2 + k + \frac{a_3}{\sinh^2 v} + \frac{a_4}{\cosh^2 v} \right]$$

for  $k$  a parameter. This is equivalent to D4B.

For the case  $\mathbf{a} = (0, 0, b_3, b_4)$  the Stäckel multiplier is  $b_3/(x^2 + y^2 - 1)^2 + b_4/(x^2 + y^2 + 1)^2$ . In terms of the coordinates  $x = -ie^{iu} \cosh v, y = e^{iu} \sinh v$  the Hamiltonian again takes a form equivalent to D4B.

For the case  $\mathbf{a} = (1, 1, a, a)$ , using polar coordinates as directly above, we see that the Hamiltonian takes the form

$$H = \frac{1}{\left[ \frac{1}{\sin^2 2\theta} + \frac{a}{\sinh^2 2v} \right]} \left[ \partial_\theta^2 + \partial_v^2 + \frac{a_1}{\cos^2 \theta} + \frac{a_2}{\sin^2 \theta} + \frac{a_3}{\sinh^2 v} + \frac{a_4}{\cosh^2 v} \right],$$

equivalent to D4C.

From these examples we note that it is always possible to choose coordinates for which the entire Hamiltonian is a rational function.

### 3.1.2 [1,1,1,1] to [2,1,1] contraction and Stäckel transforms

For fixed  $A_j, B_j, D_j$  we have the expansions

$$\begin{aligned} V_{[1,1,1,1]}^A &= \frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3}{x_3^2} + \frac{A_4}{x_4^2} \\ &= \frac{A_3}{x_3'^2} + \frac{A_4}{x_4'^2} + \frac{2(A_2 - A_1)\epsilon^2}{(x_1' + ix_2')^2} + \frac{4A_2(-x_1' + ix_2')\epsilon^4}{(x_1' + ix_2')^3} + O(\epsilon^6), \\ V_{[2,1,1]}^A &= \frac{A_1}{(x_1 + ix_2)^2} + \frac{A_2(x_1 - ix_2)}{(x_1 + ix_2)^2} + \frac{A_3}{x_3^2} + \frac{A_4}{x_4^2} \\ &= \frac{A_3}{x_3'^2} + \frac{A_4}{x_4'^2} - \frac{A_1}{2(x_1' + ix_2')^2} \epsilon^2 + \frac{(A_2 + 2A_1)(x_1' - ix_2')}{4(x_1' + ix_2')^3} \epsilon^4 + O(\epsilon^6), \\ V_{[2,2]}^A &= \frac{A_1}{(x_1 + ix_2)^2} + \frac{A_2(x_1 - ix_2)}{(x_1 + ix_2)^3} + \frac{A_3}{(x_3 + ix_4)^2} + \frac{A_4(x_3 - ix_4)}{(x_3 + ix_4)^3} \\ &= \frac{A_3}{(x_3' + ix_4')^2} + \frac{A_4(x_3' - ix_4')}{(x_3' + ix_4')^3} - \frac{A_1}{2(x_1' + ix_2')^2} \epsilon^2 \\ &\quad + \frac{(A_2 + 2A_1)(x_1' - ix_2')}{4(x_1' + ix_2')^3} \epsilon^4 + O(\epsilon^6), \\ V_{[3,1]}^B &= \frac{B_1}{(x_1 + ix_2)^2} + \frac{B_2 x_1}{(x_1 + ix_2)^3} + \frac{B_3(4x_3^2 + x_4^2)}{(x_1 + ix_2)^4} + \frac{B_4}{x_4^2} \\ &= \frac{B_3(4x_3'^2 + x_4'^2)}{(x_1' + ix_2')^4} + \frac{B_4}{x_4'^2} - \frac{(B_2 + 2B_1)}{4(x_1' + ix_2')^2} \epsilon^2 + O(\epsilon^4), \\ V_{[4]}^D &= -\frac{D_1}{2(x_1' + ix_2')^2} \epsilon^2 + \frac{i\sqrt{2}}{4} \frac{D_2(x_3' + ix_4') - 2D_3(x_3' - ix_4')}{(x_1' + ix_2')^3} \epsilon^3 \\ &\quad + \left[ \frac{3D_3(x_3' + ix_4')^2}{(x_1' + ix_2')^4} - \frac{(D_1 + 2D_4)(x_3'^2 + x_4'^2)}{2(x_1' + ix_2')^4} \right] \epsilon^4 + O(\epsilon^5), \text{ (see (49))}, \end{aligned}$$

### 3.1.3 Conformal Stäckel transforms of the [2,1,1] system

We write the potential in the normalized form

$$V'_{[2,1,1]} = \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3(x_3 - ix_4)}{(x_3 + ix_4)^3} + \frac{a_4}{(x_3 + ix_4)^2}, \quad (40)$$

and designate it via the vector  $(a_1, a_2, a_3, a_4)$ .

1. The potential  $(1, 1, 0, 0)$  generates a conformal Stäckel transform to  $S4$ .
2. The potentials  $(1, 0, 0, 0)$ ,  $(0, 1, 0, 0)$  generate conformal Stäckel transforms to  $S2$ .
3. The potential  $(0, 0, 0, 1)$  generates a conformal Stäckel transforms to  $E1$ .
4. The potential  $(0, 0, 1, 0)$  generates a conformal Stäckel transform to  $E16$ .
5. Potentials  $(a_1, a_2, 0, 0)$ , with  $a_1 a_2 \neq 0$ ,  $a_1 \neq a_2$  generate conformal Stäckel transforms to  $D4A$ .
6. Potentials  $(0, 0, a_3, a_4)$ , with  $a_3 a_4 \neq 0$  generate conformal Stäckel transforms to  $D3B$ .
7. Potentials  $(a, 0, 0, 1)$  and  $(0, a, 0, 1)$  with  $a \neq 0$  generate conformal Stäckel transforms to  $D2B$ .
8. Potentials  $(1, 1, a, 0)$  with  $a \neq 0$  generate conformal Stäckel transforms to  $D2C$ .
9. Each potential not proportional to one of these must generate a conformal Stäckel transform to a superintegrable system on a Koenigs space in the family  $K[2, 1, 1]$ .

**Basis of conformal symmetries for original system:** Let  $H_0 = \sum_{j=1}^4 \partial_{x_j}^2$ . A basis is

$$H_0 + V_{[1,1,1,1]}, \quad Q_{12}, \quad Q_{13},$$

where

$$Q_{jk} = L_{jk}^2 + a_j \frac{x_k^2}{x_j^2} + a_k \frac{x_j^2}{x_k^2}, \quad 1 \leq j < k \leq 4.$$

**Contraction of basis:** Using the notation of (39) we have

$$\begin{aligned} H_0 + V_{[1,1,1,1]} &\rightarrow H'_0 + V_{[2,1,1]}, \\ Q'_{12} &= Q_{12} - \frac{b_1}{2\epsilon^2} - \frac{b_2}{2\epsilon^4} = (L'_{12})^2 + b_1 \left( \frac{x'_1 - ix'_2}{x'_1 + ix'_2} \right) + b_2 \left( \frac{x'_1 - ix'_2}{x'_1 + ix'_2} \right)^2, \\ Q'_{13} &= 2\epsilon^2 Q_{13} = (L'_{23} - iL'_{13})^2 + \frac{b_2 x_3'^2}{(x'_1 + ix'_2)^2} - \frac{b_3 (x'_1 + ix'_2)^2}{x_3'^2}, \end{aligned}$$

If we apply the same  $[1, 1, 1, 1] \rightarrow [2, 1, 1]$  contraction to the  $[2, 1, 1]$  system, the system contracts to itself, but with parameters  $c_1, \dots, c_4$  where

$$b_1 = -\frac{2c_1}{\epsilon^2}, \quad b_2 = \frac{c_1}{\epsilon^2} + \frac{4c_2}{\epsilon^4}, \quad b_3 = c_3, \quad b_4 = c_4.$$

If we apply the same contraction to the  $[2, 2]$  system, the system contracts to itself, but with altered parameters, and to  $[0]$ .

If we apply the same contraction to the  $[3, 1]$  system, the system contracts to (1) or to itself.

If we apply the same contraction to the [4] system the system contracts to (2) or to a system with potential

$$V[0] = \frac{c_1}{(x'_1 + ix'_2)^2} + \frac{c_2x'_3 + c_3x'_4}{(x'_1 + ix'_2)^3} + c_4 \frac{x'^2_3 + x'^2_4}{(x'_1 + ix'_2)^4}. \quad (41)$$

If we apply this same contraction to the [0] system, (42) it contracts to itself with altered parameters.

If we apply this same contraction to the (1) system, (57) it contracts to (2) or to itself with altered parameters.

If we apply this same contraction to the (2), (58) it contracts to itself with altered parameters.

### 3.1.4 Conformal Stäckel transforms of the [0] system

We write the potential  $V[0]$  in the normalized form

$$V'_{[0]} = \frac{c_1}{(x_3 + ix_4)^2} + \frac{c_2x_1 + c_3x_2}{(x_3 + ix_4)^3} + c_4 \frac{x^2_1 + x^2_2}{(x_3 + ix_4)^4}, \quad (42)$$

and designate it by the vector  $(c_1, c_2, c_3, c_4)$ .

1. The potentials  $(\frac{c^2_2 + c^2_3}{4}, c_2, c_3, 1)$  generate conformal Stäckel transforms to  $E20$ .
2. The potentials  $(c_1, 1, \pm i, 0)$  generate conformal Stäckel transforms to  $E11$ .
3. The potential  $(1, 0, 0, 0)$  generates a conformal Stäckel transform to  $E3'$ .
4. Potentials  $(c_1, c_2, c_3, 0)$ , with  $c^2_2 + c^2_3 \neq 0$  generate conformal Stäckel transforms to  $D1C$ .
5. Potentials  $(c_1, c_2, c_3, 1)$ , with  $c_1 \neq \frac{c^2_2 + c^2_3}{4}$  generate conformal Stäckel transforms to  $D3A$ .
6. Each potential not proportional to one of these must generate a conformal Stäckel transform to a superintegrable system on a Koenigs space in the family  $K[0]$ .

## 3.2 [1,1,1,1] to [2,2]:

$$\begin{aligned} L'_{12} &= L_{12}, \quad L'_{34} = L_{34}, \quad L'_{24} + L'_{13} = L_{24} + L_{13}, \\ L'_{24} - L'_{13} &= (\epsilon^2 + \frac{1}{\epsilon^2})L_{13} - \frac{1}{\epsilon^2}(iL_{14} - L_{24} - iL_{23}), \\ L'_{23} - L'_{14} &= 2L_{23} + iL_{13} - iL_{24}, \\ L'_{23} + L'_{14} &= i \left( (\epsilon^2 - \frac{1}{\epsilon^2})L_{13} + \frac{1}{\epsilon^2}(iL_{14} + L_{24} + iL_{23}) \right). \end{aligned}$$

Coordinate implementation

$$\begin{aligned} x_1 &= \frac{i}{\sqrt{2}\epsilon}(x'_1 + ix'_2), \quad x_2 = \frac{1}{\sqrt{2}} \left( \frac{x'_1 + ix'_2}{\epsilon} + \epsilon(x'_1 - ix'_2) \right), \\ x_3 &= \frac{i}{\sqrt{2}\epsilon}(x'_3 + ix'_4), \quad x_4 = \frac{1}{\sqrt{2}} \left( \frac{x'_3 + ix'_4}{\epsilon} + \epsilon(x'_3 - ix'_4) \right), \end{aligned}$$

**Limit of 2D potential:**

$$V_{[1,1,1,1]} \xrightarrow{\epsilon \rightarrow 0} V_{[2,2]}$$

where

$$V_{[2,2]} = \frac{b_1}{(x'_1 + ix'_2)^2} + \frac{b_2(x'_1 - ix'_2)}{(x'_1 + ix'_2)^3} + \frac{b_3}{(x'_3 + ix'_4)^2} + \frac{b_4(x'_3 - ix'_4)}{(x'_3 + ix'_4)^3}, \quad (43)$$

and

$$a_1 = -\frac{1}{2} \frac{b_1}{\epsilon^2} - \frac{b_2}{4\epsilon^4}, \quad a_2 = -\frac{b_2}{4\epsilon^4}, \quad a_3 = -\frac{1}{2} \frac{b_3}{\epsilon^2} - \frac{b_4}{4\epsilon^4}, \quad a_4 = -\frac{b_4}{4\epsilon^4}.$$

### 3.2.1 Conformal Stäckel transforms of the [2,2] system

We designate the potential (43) by the vector  $(b_1, b_2, b_3, b_4)$ .

1. The potential  $(0, 0, 1, 0)$  generates a conformal Stäckel transform to  $E8$ .
2. The potential  $(0, 0, 0, 1)$  generates a conformal Stäckel transform to  $E17$ .
3. Potentials  $(1, 0, a, 0)$  generate conformal Stäckel transforms to  $E7$ .
4. Potentials  $(0, 1, 0, a)$  generate conformal Stäckel transforms to  $E19$ .
5. Potentials  $(0, 0, b_3, b_4)$ , with  $b_3 b_4 \neq 0$  generate conformal Stäckel transforms to  $D3C$ .
6. Potentials  $(b_1, b_2, 0, 0)$  with  $b_1 b_2 \neq 0$  generate conformal Stäckel transforms to  $D3D$ .
7. Each potential not proportional to one of these must generate a conformal Stäckel transform to a superintegrable system on a Koenigs space in the family  $K[2, 2]$ .

**Contracted basis:**

$$H_0 + V_{[1,1,1,1]} \rightarrow H'_0 + V_{[2,2]},$$

$$Q_{12} - \frac{b_2}{2\epsilon^4} - \frac{b_1}{2\epsilon^2} \rightarrow Q'_1 = L'^2_{12} + b_1 \frac{x'_1 - ix'_2}{x'_1 + ix'_2} + b_2 \frac{(x'_1 - ix'_2)^2}{(x'_1 + ix'_2)^2},$$

$$4\epsilon^4 Q_{13} \rightarrow Q'_2 = (L'_{13} + iL'_{14} + iL'_{23} - L'_{24})^2 - b_2 \frac{(x'_3 + ix'_4)^2}{(x'_1 + ix'_2)^2} - b_4 \frac{(x'_1 + ix'_2)^2}{(x'_3 + ix'_4)^2},$$

Note also that

$$\epsilon^2 (Q_{23} - Q_{14}) \rightarrow Q'_3 = -\frac{i}{2} \{L'_{14} - L'_{23}, iL'_{23} + L'_{13} - L'_{24} + iL'_{14}\} - \frac{b_1}{2} \frac{(x'_3 + ix'_4)^2}{(x'_1 + ix'_2)^2}$$

$$- b_2 \frac{(x'_2 x'_4 + x'_1 x'_3)(x'_3 + ix'_4)}{(x'_1 + ix'_2)^3} + \frac{b_3}{2} \frac{(x'_1 + ix'_2)^2}{(x'_3 + ix'_4)^2} + b_4 \frac{(x'_2 x'_4 + x'_1 x'_3)(x'_1 + ix'_2)}{(x'_3 + ix'_4)^3}$$

If we apply the same  $[1, 1, 1, 1] \rightarrow [2, 2]$  contraction to the  $[2, 1, 1]$  system with potential parameters  $k_1, \dots, k_4$ , the system contracts to the  $[2, 2]$  potential with parameters  $b_1, \dots, b_4$ , where,

$$k_1 = -\frac{2b_1}{\epsilon^2}, \quad k_2 = \frac{4b_3}{\epsilon^4}, \quad k_3 = -\frac{b_2}{2\epsilon^2} - \frac{b_4}{4\epsilon^4}, \quad k_4 = -\frac{b_4}{4\epsilon^4},$$

or to a special case of  $E15$ .

If we apply the same contraction to the  $[2, 2]$  system we recover the same system but with altered parameters, or  $[0]$ . If we apply the same contraction to the superintegrable  $[3, 1]$  system in the form

$$V[3, 1]' = \frac{k_1}{(x_1 + ix_2)^2} + \frac{k_4 x_3}{(x_1 + ix_2)^3} + k_3 \frac{(4x_3^2 + x_4^2)}{(x_1 + ix_2)^4} + \frac{k_4}{x_4^2},$$



the system contracts to a special case of  $E15$ , or to one with potential of the form

$$V(1) = \frac{c_1}{(x'_1 + ix'_2)^2} + \frac{c_2}{(x'_3 + ix'_4)^2} + c_3 \frac{x'_3 + ix'_4}{(x'_1 + ix'_2)^3} + c_4 \frac{(x'_3 + ix'_4)^2}{(x'_1 + ix'_2)^4}. \quad (44)$$

It admits 2 1st order symmetries and is Stäckel equivalent to special cases of the Euclidean superintegrable system  $E15$  via transforms  $(x'_1 + ix'_2)^2$  or  $(x'_3 + ix'_4)^2$ .

If we apply the same contraction to the superintegrable [4] system we get system conformally equivalent to (58). This admits a 1st order symmetry and goes to a special case of  $E15$  by a conformal Stäckel transform.

If we apply this same contraction to the [0] system, (42) it contracts to itself with altered parameters.

If we apply this same contraction to the (1) system, (57) it contracts to itself with altered parameters, or to a special case of  $E15$ .

If we apply this same contraction to the (2) system, (58) it contracts to itself with altered parameters.

### 3.2.2 [1,1,1,1] to [2,2] contraction and Stäckel transforms

For fixed  $A_j$  we have the expansions

$$\begin{aligned} V_{[1,1,1,1]}^A &= \frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3}{x_3^2} + \frac{A_4}{x_4^2} \\ &= \frac{2(A_2 + A_4 - A_1 - A_3)\epsilon^2}{(x'_1 + ix'_2)^2} + \left( \frac{4A_4(-x'_3 + ix'_4)}{(x'_3 + ix'_4)^3} + \frac{4A_2(-x'_1 + ix'_2)}{(x'_1 + ix'_2)^3} \right) \epsilon^4 \\ &\quad + \left( \frac{6A_4(-x'_3 + ix'_4)^2}{(x'_3 + ix'_4)^4} + \frac{6A_2(-x'_1 + ix'_2)^2}{(x'_1 + ix'_2)^4} \right) \epsilon^6 + O(\epsilon^8), \\ V_{[2,1,1]}^A &= \frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3(x_3 - ix_4)}{(x_3 + ix_4)^3} + \frac{A_4}{(x_3 + ix_4)^2} \\ &= \left( \frac{2(A_2 - A_1)}{(x'_1 + ix'_2)^2} - \frac{A_4}{2(x'_3 + ix'_4)^2} \right) \epsilon^2 \\ &\quad + \left( -\frac{4A_2(x'_1 - ix'_2)}{(x'_1 + ix'_2)^3} + \frac{(A_3 + 2A_4)(x'_3 - ix'_4)}{4(x'_3 + ix'_4)^3} \right) \epsilon^4 + O(\epsilon^6), \\ V_{[2,2]}^A &= \frac{A_1}{(x_1 + ix_2)^2} + \frac{A_2(x_1 - ix_2)}{(x_1 + ix_2)^3} + \frac{A_3}{(x_3 + ix_4)^2} + \frac{A_4(x_3 - ix_4)}{(x_3 + ix_4)^3} \\ &= -\frac{1}{2} \left( \frac{A_1}{(x'_1 + ix'_2)^2} + \frac{A_3}{(x'_3 + ix'_4)^2} \right) \epsilon^2 \\ &\quad + \frac{1}{4} \left( \frac{(A_2 + 2A_1)(x'_1 - ix'_2)}{(x'_1 + ix'_2)^3} + \frac{(A_4 + 2A_3)(x'_3 - ix'_4)}{(x'_3 + ix'_4)^3} \right) \epsilon^4 + O(\epsilon^6), \end{aligned}$$

### 3.3 [2,1,1] to [3,1]

Coordinate implementation:

$$\begin{aligned} x_1 + ix_2 &= -\frac{i\sqrt{2}\epsilon}{2}x'_2 + \frac{(ix'_1 - x'_3)}{\epsilon}, \\ x_1 - ix_2 &= -\epsilon(x'_3 + ix'_1) + \frac{3i\sqrt{2}x'_2}{4\epsilon} + \frac{1}{2}\frac{(ix'_1 - x'_3)}{\epsilon^3}, \\ x_3 &= -\frac{1}{2}x'_2 - \frac{\sqrt{2}}{2}\frac{(x'_1 + ix'_3)}{\epsilon^2}, \quad x_4 = x'_4. \end{aligned}$$

$$\begin{aligned}
L'_{24} &= \frac{\sqrt{2}i}{2\epsilon}(L_{14} + iL_{24}) - L_{34}, \quad L'_{14} + iL'_{34} = -i\epsilon(L_{14} + iL_{24}), \\
L'_{14} - iL'_{34} &= \frac{1}{\epsilon} \left( iL_{14}(1 + \frac{1}{2\epsilon^2}) + L_{24}(1 - \frac{1}{2\epsilon^2}) - \frac{\sqrt{2}}{\epsilon}L_{34} \right), \\
L'_{13} &= -L_{12} - 2\sqrt{2}L_{13}(\epsilon + 2\epsilon^3), \\
L'_{23} + iL'_{12} &= 4\epsilon^3L_{13}, \quad L'_{23} - iL'_{12} = (2\sqrt{2} - \frac{\sqrt{2}}{\epsilon^2})L_{12} \\
&\quad + (8\epsilon^3 + 4\epsilon - \frac{2}{\epsilon} + \frac{1}{2\epsilon^3})L_{13} + \frac{i}{2\epsilon^3}L_{23}.
\end{aligned}$$

**Limit of 2D potential:**

$$V_{[2,1,1]} \xrightarrow{\epsilon \rightarrow 0} V_{[3,1]},$$

where

$$V_{[3,1]} = \frac{c_1}{(x'_1 + ix'_3)^2} + \frac{c_2x'_2}{(x'_1 + ix'_3)^3} + \frac{c_3(4x'^2_2 + x'^2_4)}{(x'_1 + ix'_3)^4} + \frac{c_4}{x'^2_4}, \quad (45)$$

and

$$b_1 = \frac{c_3}{\epsilon^6} + \frac{\sqrt{2}c_2}{4\epsilon^4} - \frac{c_1}{\epsilon^2}, \quad b_2 = -\frac{c_3}{\epsilon^4} - \frac{\sqrt{2}c_2}{2\epsilon^2}, \quad b_3 = \frac{c_3}{4\epsilon^8}, \quad b_4 = c_4.$$

### 3.3.1 Conformal Stäckel transforms of the [3,1] system

We write potential  $V_{[3,1]}$  in the normalized form

$$V'_{[3,1]} = \frac{a_1}{(x_3 + ix_4)^2} + \frac{a_2x_1}{(x_3 + ix_4)^3} + \frac{a_3(4x_1^2 + x_2^2)}{(x_3 + ix_4)^4} + \frac{a_4}{x_2^2}, \quad (46)$$

and designate it  $(a_1, a_2, a_3, a_4)$ .

1. The potential  $(0, 0, 0, 1)$  generates a conformal Stäckel transform to  $S1$ .
2. The potential  $(1, 0, 0, 0)$  generates a conformal Stäckel transform to  $E2$ .
3. The potential  $(a, 1, 0, 0)$  generates a conformal Stäckel transform to  $D1B$ .
4. The potential  $(0, 0, 1, 0)$  generates a conformal Stäckel transform to  $D2A$ .
5. Each potential not proportional to one of these must generate a conformal Stäckel transform to a superintegrable system on a Koenigs space in the family  $K[3, 1]$ .

**Basis of conformal symmetries for original system:**

$$\begin{aligned}
H_0 + V_{[2,1,1]}, \\
Q_{12} &= (L_{12})^2 + b_1 \left( \frac{x_1 - ix_2}{x_1 + ix_2} \right) + b_2 \left( \frac{x_1 - ix_2}{x_1 + ix_2} \right)^2, \\
Q_{13} &= (L_{23} - iL_{13})^2 + \frac{b_2x_3^2}{(x_1 + ix_2)^2} - \frac{b_3(x_1 + ix_2)^2}{x_3^2},
\end{aligned}$$

**Contraction of basis:**

$$H_0 + V_{[2,1,1]} \rightarrow H'_0 + V_{[3,1]},$$

$$\begin{aligned} Q'_{12} &= -2\epsilon^4 Q_{12} + \frac{c_3}{2\epsilon^4} - c_1 = (L'_{12} - iL'_{23})^2 + \frac{c_2 x'_2}{x'_1 + ix'_3} + \frac{4c_3 x'^2_2}{(x'_1 + ix'_3)^2}, \\ Q'_{13} &= -\frac{\sqrt{2}}{4}(Q_{13} + 2\epsilon^2 Q_{12} - \frac{3c_3}{2\epsilon^6} - \frac{\sqrt{2} c_2}{4\epsilon^4} + c_1) = \\ &= \frac{1}{2}\{L'_{13}, L'_{23} + iL'_{12}\} + \frac{c_1 x'_2}{x'_1 + ix'_3} + \frac{c_2(x'^2_4 + 4x'^2_2)}{4(x'_1 + ix'_3)^2} + \frac{2c_3 x'_2(x'^2_4 + 2x'^2_2)}{(x'_1 + ix'_3)^3}, \end{aligned}$$

If we apply the same  $[2, 1, 1] \rightarrow [3, 1]$  contraction to the  $[1, 1, 1, 1]$  system, the system contracts to the  $[3, 1]$  potential, but with parameters  $c_1, \dots, c_4$  where

$$a_1 = \frac{c_1}{\epsilon^8} + c_2 \left( \frac{16}{\epsilon^{10}} + \frac{1}{\epsilon^{12}} \right), \quad a_2 = \frac{c_2}{\epsilon^{12}}, \quad a_3 = \frac{c_3}{\epsilon^4} + \frac{8c_1 - 512c_2}{\epsilon^6} + \frac{64c_2}{\epsilon^8}, \quad a_4 = c_4.$$

If we apply the same contraction to the  $[2, 2]$  system, the system contracts to one with potential

$$V = \frac{c_1}{(x'_1 + ix'_3)^2} + \frac{c_2 x'_2 + c_3 x'_4}{(x'_1 + ix'_3)^3} + c_4 \frac{x'^2_2 + x'^2_4}{(x'_1 + ix'_3)^4}, \quad (47)$$

where

$$\begin{aligned} b_1 &= -\frac{\sqrt{2} c_2}{4\epsilon^4} + \frac{c_4}{4\epsilon^6}, \quad b_2 = -\frac{c_4}{4\epsilon^4}, \quad b_3 = \sqrt{2} \frac{(-2c_2 + ic_3)}{8\epsilon^6} + \frac{c_4}{8\epsilon^8}, \\ b_4 &= \frac{c_1}{2\epsilon^4} + \sqrt{2} \frac{(-ic_3 + c_2)}{8\epsilon^6} - \frac{c_4}{16\epsilon^8}. \end{aligned}$$

This is conformally equivalent to (42).

If we apply this same contraction to the system with  $V[3, 1]$  potential, the system contracts to one with  $V[3, 1]$  potential again, but with different parameters, or to  $[0]$ .

If we apply this same contraction to the system with  $V[4]$  potential, the system contracts to one with potential (47) again, but with different parameters.

If we apply this same contraction to the  $[0]$  system, (47) it contracts to itself with altered parameters.

If we apply this same contraction to the (1) system, (57) it becomes a potential conformally equivalent to (58).

If we apply this same contraction to the (2) system, (58) it contracts to itself with altered parameters.

### 3.3.2 $[2,1,1]$ to $[3,1]$ contraction and Stäckel transforms

For fixed  $A_j, B_j$  we have the expansion

$$\begin{aligned} V_{[1,1,1,1]}^A &= \frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3}{x_3^2} + \frac{A_4}{x_4^2} = \\ &= \frac{A_4}{x'^2_4} + \frac{2A_3}{(x'_1 + ix'_3)^2} \epsilon^4 + \left( \frac{16(A_2 - A_1)}{(x'_1 + ix'_3)^2} - \frac{2\sqrt{2}A_3 x'_2}{(x'_1 + ix'_3)^3} \right) \epsilon^6 + O(\epsilon^8). \\ V_{[2,1,1]}^A &= \frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3(x_3 - ix_4)}{(x_3 + ix_4)^3} + \frac{A_4}{(x_3 + ix_4)^2} = \frac{2(A_3 + A_4)}{(x'_1 + ix'_3)^2} \epsilon^4 \end{aligned}$$

$$\begin{aligned}
& + \left( \frac{16(A_2 - A_1)}{(x'_1 + ix'_3)^2} + \frac{(3A_3 + 2A_4)\sqrt{2}(-x'_2 + 2ix'_4)}{(x'_1 + ix'_3)^3} + \frac{A_3\sqrt{2}(x'_2 + 2ix'_4)}{(x'_1 + ix'_3)^3} \right) \epsilon^6 + O(\epsilon^8). \\
V_{[2,2]}^A &= \frac{A_1}{(x_1 + ix_2)^2} + \frac{A_2(x_1 - ix_2)}{(x_1 + ix_2)^3} + \frac{A_3}{(x_3 + ix_4)^2} + \frac{A_4(x_3 - ix_4)}{(x_3 + ix_4)^3} \\
&= -\frac{A_2}{2(x'_1 + ix'_3)^2} + \left( -\frac{A_1}{(x'_1 + ix'_3)^2} - \frac{3\sqrt{2}A_2x'_2}{(x'_1 + ix'_2)^3} \right) \epsilon^2 \\
&\quad \left( -\frac{\sqrt{2}A_1x'_2}{(x'_1 + ix'_3)^3} - \frac{(4x_4'^2 + 19x_2'^2)A_2}{(x'_1 + ix'_3)^4} \right) \epsilon^4 + O(\epsilon^6), \\
V_{[3,1]}^B &= \frac{B_1}{(x_1 + ix_2)^2} + \frac{B_2x_2}{(x_1 + ix_3)^3} + \frac{B_3(4x_2^2 + x_4^2)}{(x_1 + ix_3)^4} + \frac{B_4}{x_4^2} \\
&= \frac{B_4}{x_4'^2} - 16\frac{(B_1 + iB_2 - 4B_3)}{(x'_1 + ix'_3)^2} \epsilon^6 + O(\epsilon^7),
\end{aligned}$$

### 3.4 [1,1,1,1] to [4]:

In this case there is a 2-parameter family of contractions, but all lead to the same result. Let  $A, B$  be constants such that  $AB(1-A)(1-B)(A-B) \neq 0$ . Coordinate implementation

$$\begin{aligned}
x_1 &= \frac{i}{\sqrt{2AB}} \epsilon^3 (x'_1 + ix'_2), \\
x_2 &= \frac{(x'_1 + ix'_2) + \epsilon^2(x'_3 + ix'_4) + \epsilon^4(x'_3 - ix'_4) + \epsilon^6(x'_1 - ix'_2)}{\sqrt{2(A-1)(B-1)} \epsilon^3}, \\
x_3 &= \frac{(x'_1 + ix'_2) + A\epsilon^2(x'_3 + ix'_4) + A^2\epsilon^4(x'_3 - ix'_4) + A^3\epsilon^6(x'_1 - ix'_2)}{\sqrt{2A(A-1)(A-B)} \epsilon^3}, \\
x_4 &= \frac{(x'_1 + ix'_2) + B\epsilon^2(x'_3 + ix'_4) + B^2\epsilon^4(x'_3 - ix'_4) + B^3\epsilon^6(x'_1 - ix'_2)}{\sqrt{2B(B-1)(B-A)} \epsilon^3},
\end{aligned}$$

In this case:

$$\begin{aligned}
iL'_{14} + iL'_{23} + L'_{13} - L'_{24} &= -2i\epsilon^4 \sqrt{AB(A-1)(B-1)} L_{12}, \\
iL'_{14} - iL'_{23} - L'_{13} - L'_{24} &= 2i\epsilon^2 \left( \sqrt{B(A-1)(A-B)} L_{13} - \sqrt{AB(A-1)(B-1)} L_{12} \right), \\
L'_{12} &= \frac{\sqrt{AB}}{\sqrt{(A-1)(B-1)}} L_{12} + \frac{\sqrt{B}}{\sqrt{(A-1)(A-B)}} L_{13} \\
&\quad - \frac{i\sqrt{A}}{\sqrt{(B-1)(A-B)}} L_{14}, \\
L'_{34} &= \frac{\sqrt{B(B-1)}}{\sqrt{A(A-1)}} L_{12} - \frac{\sqrt{B(A-B)}}{\sqrt{(A-1)}} L_{13} + i \frac{\sqrt{(B-1)(A-B)}}{\sqrt{A}} L_{23}, \\
-iL'_{14} + iL'_{23} - L'_{13} - L'_{24} &= \frac{2}{\epsilon^2} \left( \frac{i(A+B-1)}{\sqrt{AB(A-1)(B-1)}} L_{12} + \frac{i\sqrt{B}}{\sqrt{(A-1)(A-B)}} L_{13} \right. \\
&\quad - \frac{\sqrt{A}}{\sqrt{B(B-1)(A-B)}} L_{14} + \frac{\sqrt{(B-1)}}{\sqrt{A(A-B)}} L_{23} \\
&\quad \left. - \frac{i\sqrt{(A-1)}}{\sqrt{B(A-B)}} L_{24} \right), \\
iL'_{14} + iL'_{23} - L'_{13} + L'_{24} &= \frac{2i}{\epsilon^4} \left( -\frac{1}{\sqrt{AB(A-1)(B-1)}} (L_{12} + L_{34}) \right. \\
&\quad + \frac{i}{\sqrt{A(B-1)(A-B)}} (L_{14} + L_{23}) \\
&\quad \left. - \frac{1}{\sqrt{B(A-1)(A-B)}} (L_{13} - L_{24}) \right).
\end{aligned} \tag{48}$$

**Limit of 2D potential:**

$$V_{[1,1,1,1]} \xrightarrow{\epsilon \rightarrow 0} V_{[4]},$$

where

$$\begin{aligned}
V_{[4]} &= \frac{d_1}{(x'_1 + ix'_2)^2} + \frac{d_2(x'_3 + ix'_4)}{(x'_1 + ix'_2)^3} \\
&\quad + d_3 \left( \frac{3(x'_3 + ix'_4)^2}{(x'_1 + ix'_2)^4} - 2 \frac{(x'_1 + ix'_2)(x'_3 - ix'_4)}{(x'_1 + ix'_2)^4} \right) + \\
&\quad d_4 \frac{4(x'_1 + ix'_2)(x_1'^2 + x_2'^2) + 2(x'_3 + ix'_4)^3}{(x'_1 + ix'_2)^5}.
\end{aligned} \tag{49}$$

and

$$\begin{aligned}
a_1 &= -\frac{d_4}{4A^2B^2\epsilon^{12}} - \frac{d_3}{2AB^2\epsilon^{10}} - \frac{d_2}{4AB\epsilon^8} - \frac{d_1}{2AB\epsilon^6}, \\
a_2 &= -\frac{d_4}{4(1-A)^2(1-B)^2\epsilon^{12}} + \frac{d_3}{2(1-A)(1-B)^2\epsilon^{10}} - \frac{d_2}{4(1-A)(1-B)\epsilon^8}, \\
a_3 &= -\frac{d_4}{4A^2(1-A)^2(A-B)^2\epsilon^{12}}, \\
a_4 &= -\frac{d_4}{4B^2(1-B)^2(A-B)^2\epsilon^{12}} - \frac{d_3}{2B^2(1-A)^2(A-B)\epsilon^{10}}.
\end{aligned}$$

### 3.4.1 Conformal Stäckel transforms of the [4] system

We write potential  $V_{[4]}$  in the normalized form

$$V'_{[4]} = \frac{a_1}{(x_3 + ix_4)^2} + a_2 \frac{x_1 + ix_2}{(x_3 + ix_4)^3} + a_3 \frac{3(x_1 + ix_2)^2 - 2(x_3 + ix_4)(x_1 - ix_2)}{(x_3 + ix_4)^4} + a_4 \frac{4(x_3 + ix_4)(x_3^2 + x_4^2) + 2(x_1 + ix_2)^3}{(x_3 + ix_4)^5}, \quad (50)$$

and designate it  $(a_1, a_2, a_3, a_4)$ .

1. The potentials  $(1, a, 0, 0)$  generate conformal Stäckel transforms to  $E_{10}$ .
2. The potential  $(0, 1, 0, 0)$  generates a conformal Stäckel transform to  $E_9$ .
3. The potential  $(0, 0, 0, 1)$  generates a conformal Stäckel transform to  $D_{1A}$ .
4. Each potential not proportional to one of these must generate a conformal Stäckel transform to a superintegrable system on a Koenigs space in the family  $K[4]$ .

In these coordinates a basis for the conformal symmetry algebra is  $H, Q_1, Q_2$  where

$$Q_1 = \frac{1}{4}(L_{14} + L_{23} - iL_{13} + iL_{24})^2 + 4a_3 \left( \frac{x_1 + ix_2}{x_3 + ix_4} \right) + 4a_4 \left( \frac{x_1 + ix_2}{x_3 + ix_4} \right)^2, \\ Q_2 = \frac{1}{2}\{L_{23} + L_{14} - iL_{13} + iL_{24}, L_{12} + L_{34}\} + \frac{1}{4}(L_{14} - L_{23} + iL_{13} + iL_{24})^2 \\ + 2a_1 \left( \frac{x_1 + ix_2}{x_3 + ix_4} \right) + a_2 \left( 2 \frac{x_1 - ix_2}{x_3 + ix_4} - \left( \frac{x_1 + ix_2}{x_3 + ix_4} \right)^2 \right) \\ + 2a_3 \left( 6 \left( \frac{x_1^2 + x_2^2}{(x_3 + ix_4)^2} \right) - \left( \frac{x_1 + ix_2}{x_3 + ix_4} \right)^3 \right) \\ - 4a_4 \left( \left( \frac{x_1 - ix_2}{x_3 + ix_4} \right)^2 - 3 \left( \frac{(x_1 + ix_2)^2 (x_1 - ix_2)}{(x_3 + ix_4)^3} \right) + \frac{1}{4} \left( \frac{x_1 + ix_2}{x_3 + ix_4} \right)^4 \right).$$

**Basis of conformal symmetries for original system:**

$$H_0 + V_{[1,1,1,1]}, \quad Q_{12}, \quad Q_{13},$$

where

$$Q_{jk} = (x_j \partial_{x_k} - x_k \partial_{x_j})^2 + a_j \frac{x_k^2}{x_j^2} + a_k \frac{x_j^2}{x_k^2}, \quad 1 \leq j < k \leq 4.$$

**Contraction of basis:**

$$H_0 + V_{[1,1,1,1]} \rightarrow H'_0 + V_{[4]},$$

$$\epsilon^8 Q_{12} \sim \frac{-1}{4(A-1)(B-1)AB} (L'_{13} - L'_{24} + iL'_{23} + iL'_{14})^2 \\ + \frac{4d_3(x'_3 + ix'_4)}{AB(A-1)(B-1)(x'_1 + ix'_2)} \\ + \frac{d_4}{4AB(A-1)(B-1)} \left[ \frac{(x'_3 + ix'_4)^2}{(x'_1 + ix'_2)^2} + 2 \frac{x'_3 - ix'_4}{x'_1 + ix'_2} \right],$$

In this case we do not obtain a basis of symmetries for the [4] system. The basis can be computed from the contracted potential.

If we apply the same [1, 1, 1, 1]  $\rightarrow$  [4] contraction to the [2, 1, 1] system, the system contracts to a modified [4] potential, of the form

$$\begin{aligned}\tilde{V}_{[4]} &= \frac{d'_1}{(x'_1 + ix'_2)^2} + \frac{d'_2(x'_3 + ix'_4)}{(x'_1 + ix'_2)^3} \\ &+ d'_3 \left( \frac{3(x'_3 + ix'_4)^2}{(x'_1 + ix'_2)^4} - 2\lambda \frac{(x'_1 + ix'_2)(x'_3 - ix'_4)}{(x'_1 + ix'_2)^4} \right) + \\ &d'_4 \left( \frac{4\lambda(x'_1 + ix'_2)(x'_1{}^2 + x'_2{}^2)}{(x'_1 + ix'_2)^5} + \frac{2(x'_3 + ix'_4)^3}{(x'_1 + ix'_2)^5} \right),\end{aligned}$$

where  $\lambda$  is a nonzero function of  $A$  and  $B$ . However, under an appropriate conformal transformation

$$x'_1 + ix'_2 \rightarrow \mu(x'_1 + ix'_2), \quad x'_1 - ix'_2 \rightarrow \mu^{-1}(x'_1 - ix'_2),$$

we obtain the potential  $V_{[4]}$  exactly.

If we apply the same contraction to the [2, 2] system, the system contracts to

$$V = \frac{e_1}{(x'_1 + ix'_2)^2} + e_2 \frac{(x'_3 + ix'_4)}{(x'_1 + ix'_2)^3} + e_3 \frac{(x'_3 - ix'_4)}{(x'_1 + ix'_2)^3} + e_4 \frac{(x'^2_3 + x'^2_4)}{(x'_1 + ix'_2)^4}, \quad (51)$$

conformally equivalent to (42).

If we apply the same contraction to the [3, 1] system, the system contracts to

$$\begin{aligned}V &= \frac{f_1}{(x'_1 + ix'_2)^2} + f_2 \frac{(x'_3 + ix'_4)}{(x'_1 + ix'_2)^3} \\ &+ \frac{f_3}{(x'_1 + ix'_2)^4} [3\lambda(x'_3 + ix'_4)^2 + (x'_1 + ix'_2)(x'_3 - ix'_4)] \\ &+ \frac{f_4(x'_3 + ix'_4)}{(x'_1 + ix'_2)^5} [\lambda(x'_3 + ix'_4)^2 + (x'_1 + ix'_2)(x'_3 - ix'_4)],\end{aligned}$$

where the nonzero scalar  $\lambda$  depends on the choice of  $A$  and  $B$ . It can be rescaled to any desired nonzero value by a conformal transform

$$x'_1 + ix'_2 \rightarrow \mu(x'_1 + ix'_2), \quad x'_1 - ix'_2 \rightarrow \mu^{-1}(x'_1 - ix'_2).$$

This system is conformally equivalent to (50) again.

If we apply the same contraction to the [4] system, the system contracts to one with potential (51) again, but with different parameters.

If we apply the same contraction to the [0] system (51) the system contracts to one with potential (51) again, but with different parameters.

If we apply this same contraction to the (1) system, (57) it becomes a potential conformally equivalent to (58).

If we apply this same contraction to the (2) system, (58) it contracts to itself with altered parameters.

### 3.4.2 [1,1,1,1] to [4] contraction and Stäckel transforms

For fixed  $A_j$  we have (in the special case  $A = 10, B = 5$ ) the expansions

$$V_{[1,1,1,1]}^A = \frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3}{x_3^2} + \frac{A_4}{x_4^2}$$

$$\begin{aligned}
&= \frac{4(-5A_1 + 2A_2 + 30A_4 + 3A_3)}{(x'_1 + ix'_2)^2} \epsilon^6 + \frac{16(-A_2 + 3A_3 - 75A_4)(x'_3 + ix'_4)}{(x'_1 + ix'_2)^3} \epsilon^8 + O(\epsilon^{10}). \\
V_{[2,1,1]}^A &= \frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3(x_3 - ix_4)}{(x_3 + ix_4)^3} + \frac{A_4}{(x_3 + ix_4)^2} \\
&= \frac{-\frac{4}{127}(135A_1 - 54A_2 + [110 + 2\sqrt{10}]A_4) - \frac{40}{243}(161 + 44\sqrt{10})A_3}{(x'_1 + ix'_2)^2} \epsilon^6 + O(\epsilon^8),
\end{aligned}$$

### 3.5 [2,2] to [4]:

$$\begin{aligned}
L'_{12} &= i(1 + \frac{2}{\epsilon} - \frac{1}{2\epsilon^2})L_{12} + \frac{1}{\epsilon}(1 - \frac{3}{4\epsilon} + \frac{1}{4\epsilon^2})L_{13} + \frac{i}{4\epsilon^2}(3 - \frac{1}{\epsilon})L_{14} \\
&+ \frac{i}{4\epsilon^2}(3 - \frac{1}{\epsilon})L_{23} + (3 - \epsilon + \frac{3}{4\epsilon^2} - \frac{1}{4\epsilon^3})L_{24} + i(\frac{3\epsilon}{2} - 2 + \frac{1}{\epsilon} - \frac{1}{2\epsilon^2})L_{34}, \\
L'_{12} + iL'_{24} &= \epsilon(L_{13} - iL_{14}), \\
L'_{13} + iL'_{34} &= \epsilon(L_{23} - iL_{24}), \\
L'_{14} &= (-1 + \epsilon)L_{12} + i(1 - \epsilon)L_{13} + (1 + \epsilon)L_{14}, \\
L'_{23} - L'_{14} &= -L_{14} + L_{23}, \\
L'_{13} + L'_{24} &= (\frac{1}{2} - \frac{1}{\epsilon})L_{12} + \frac{i}{\epsilon}L_{13} + \frac{1}{2}L_{14} + \frac{1}{2}L_{23} + (2 + \frac{i}{\epsilon})L_{24} + (\epsilon - \frac{1}{2} + \frac{1}{\epsilon})L_{34}.
\end{aligned} \tag{52}$$

**Coordinate implementation:**

$$\begin{aligned}
x_1 &= \frac{1}{2}(\frac{1}{\epsilon} + \frac{1}{\epsilon^2})(x'_1 - ix'_4) + \frac{\epsilon}{2}(x'_1 + ix'_4) - (1 + \frac{1}{2\epsilon})(x'_2 - ix'_3) + \frac{1}{2}(\epsilon - 1)(x'_2 + ix'_3), \\
x_2 &= \frac{i}{2}(\frac{1}{\epsilon} - \frac{1}{\epsilon^2})(x'_1 - ix'_4) - \frac{i\epsilon}{2}(x'_1 + ix'_4) - i(1 - \frac{1}{2\epsilon})(x'_2 - ix'_3) + \frac{i}{2}(\epsilon + 1)(x'_2 + ix'_3), \\
x_3 &= \frac{1}{2}(\frac{1}{\epsilon} - \frac{1}{\epsilon^2})(x'_1 - ix'_4) + (-\frac{1}{2} + \frac{1}{\epsilon})(x'_2 - ix'_3), \\
x_4 &= \frac{i}{2}(\frac{1}{\epsilon} + \frac{1}{\epsilon^2})(x'_1 - ix'_4) - i(\frac{1}{2} + \frac{1}{\epsilon})(x'_2 - ix'_3).
\end{aligned}$$

**Limit of 2D potential:**

$$V_{[2,2]} \xrightarrow{\epsilon \rightarrow 0} V'_{[4]},$$

$$\begin{aligned}
V'_{[4]} &= \frac{e_1}{(x'_1 - ix'_4)^2} + \frac{e_2(x'_2 - ix'_3)}{(x'_1 - ix'_4)^3} \\
&+ e_3 \left( \frac{3(x'_2 - ix'_3)^2}{(x'_1 - ix'_4)^4} + 2 \frac{(x'_1 - ix'_4)(x'_2 + ix'_3)}{(x'_1 - ix'_4)^4} \right) + \\
&e_4 \left( \frac{4(x'_1 - ix'_4)(x'_2{}^2 + x'_3{}^2) + 2(x'_2 - ix'_3)^3}{(x'_1 - ix'_4)^5} \right),
\end{aligned} \tag{53}$$

where

$$b_1 = \frac{e_1}{\epsilon^4} + 2\frac{e_4}{\epsilon^7}, \quad b_2 = -\frac{e_2}{4\epsilon^6} - \frac{e_3}{2\epsilon^7} - \frac{e_4}{\epsilon^8}, \quad b_3 = 2\frac{e_3}{\epsilon^6} - 2\frac{e_4}{\epsilon^7}, \quad b_4 = -\frac{e_2}{4\epsilon^6} + \frac{3e_3}{2\epsilon^7} - \frac{e_4}{\epsilon^8}.$$

This is conformally equivalent to  $V[4]$ .

**Basis of conformal symmetries for original system:**

$$H_0 + V_{[2,2]}, \quad Q_1, \quad Q_3$$

**Contraction of basis:**



$$\begin{aligned}
& H_0 + V_{[2,2]} \rightarrow H'_0 + V'_{[4]}, \\
& -4\epsilon^4(Q_1 + \frac{k_4}{\epsilon^6} - \frac{k_3}{2\epsilon^5}) \rightarrow (iL'_{13} - L'_{12} - iL'_{24} - L'_{34})^2 \\
& + k_2 + 4k_3 \frac{x'_2 - ix'_3}{x'_1 - ix'_4} - 4k_4 \frac{(x'_2 - ix'_3)^2}{(x'_1 - ix'_4)^2}, \\
& \epsilon^3(Q_3 - \frac{2k_4}{\epsilon^7} + \frac{k_3}{\epsilon^6} + \frac{k_1}{2\epsilon^4}) \rightarrow \\
& \frac{i}{2}\{L'_{23} - L'_{14}, (L'_{12} - iL'_{13} + L'_{24} + L'_{34})\} + k_1 \frac{(x'_2 - ix'_3)}{(x'_1 - ix'_4)} + k_2 \frac{(x'_2 - ix'_3)^2}{(x'_1 - ix'_4)^2} \\
& + k_3 \frac{3(x'_2 - ix'_3)^3 + 2(x'_2{}^2 + x'_3{}^2)(x'_1 - ix'_4)}{(x'_1 - ix'_4)^3} \\
& - 2k_4(x'_2 - ix'_3) \frac{(x'_2 - ix'_3)^3 + 2(x'_2{}^2 + x'_3{}^2)(x'_1 - ix'_4)}{(x'_1 - ix'_4)^4}.
\end{aligned}$$

However, the second limit here is equivalent to the contracted Hamiltonian, not an independent basis element.

If we apply the  $[2, 2] \rightarrow [4]$  contraction to the  $[1, 1, 1, 1]$  system, the system contracts to

$$\begin{aligned}
V[4]'' &= \frac{f_1}{(x'_1 - ix'_4)^2} + f_2 \frac{(x'_2 - ix'_3)}{(x'_1 - ix'_4)^3} \\
&+ \frac{f_3}{(x'_1 - ix'_4)^4} [3(x'_2 - ix'_3)^2 + 2(x'_1 - ix'_4)(x'_2 + ix'_3)] \\
&+ \frac{f_4(x'_2 - ix'_3)}{(x'_1 - ix'_4)^5} [(x'_2 - ix'_3)^2 + 2(x'_1 - ix'_4)(x'_2 + ix'_3)], \quad (54)
\end{aligned}$$

where

$$\begin{aligned}
b_1 &= \frac{f_1 + 2f_3}{4\epsilon^4} + \frac{f_2 + 10f_4}{64\epsilon^6} - \frac{f_3 - 4f_4}{32\epsilon^7} + \frac{f_4}{32\epsilon^8}, \\
b_2 &= \frac{f_2 + 10f_4}{64\epsilon^6} - \frac{f_3 + 4f_4}{32\epsilon^7} + \frac{f_4}{32\epsilon^8}, \\
b_3 &= \frac{f_2 - 16f_3 + 10f_4}{64\epsilon^6} + \frac{f_3 - 4f_4}{32\epsilon^7} + \frac{f_4}{32\epsilon^8}, \\
b_4 &= \frac{f_2 + 16f_3 + 10f_4}{64\epsilon^6} + \frac{3f_3 + 4f_4}{32\epsilon^7} + \frac{f_4}{32\epsilon^8},
\end{aligned}$$

also conformally equivalent to  $V_{[4]}$ .

If we apply the same contraction to the  $[2, 1, 1]$  system, the system contracts to potential (2), or to (54) again, except that now

$$\begin{aligned}
b_1 &= \frac{f_1 - f_3}{\epsilon^4} - \frac{2f_2 + 5f_4}{2\epsilon^5} - \frac{2f_3}{\epsilon^6} + \frac{f_4}{\epsilon^7}, \\
b_2 &= \frac{3f_3}{2\epsilon^7} - \frac{f_4}{2\epsilon^8}, \\
b_3 &= \frac{f_2 + 7f_4}{4\epsilon^5} - \frac{f_2 + 7f_4}{16\epsilon^6} + \frac{f_3 - 4f_4}{32\epsilon^7} + \frac{f_4}{32\epsilon^8}, \\
b_4 &= -\frac{f_2 + 7f_4}{4\epsilon^5} - \frac{f_2 + 7f_4}{16\epsilon^6} + \frac{f_3 + 4f_4}{32\epsilon^7} + \frac{f_4}{32\epsilon^8}.
\end{aligned}$$

If we apply the same contraction to the  $[3, 1]$  system, the system contracts to a system with potential

$$V(2) = \frac{c_1}{(x'_1 - ix'_4)^2} + \frac{c_2(x'_2 - ix'_3)}{(x'_1 - ix'_4)^3} + \frac{c_3(x'_2 - ix'_3)^2}{(x'_1 - ix'_4)^4} + \frac{c_4(x'_2 - ix'_3)^3}{(x'_1 - ix'_4)^5}. \quad (55)$$

This system admits a first order symmetry. It corresponds to a special case of the flat space superintegrable system  $E15$  via the transform  $(x'_1 - ix'_4)^2$ .

If we apply the same contraction to the [4] system, the system contracts to a system with potential (50) again, but with different parameters.

If we apply the same contraction to the [0] system (51) the system contracts to one with potential (51) again, but with different parameters. or to (2).

If we apply this same contraction to the (1) system, (57) it becomes a potential conformally equivalent to (58).

If we apply this same contraction to the (2) system, (58) it contracts to itself with altered parameters.

### 3.5.1 [2,2] to [4] contraction and Stäckel transforms

For fixed  $A_j$  we have the expansions

$$\begin{aligned}
V_{[1,1,1,1]}^A &= \frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3}{x_3^2} + \frac{A_4}{x_4^2} = 4 \frac{A_1 - A_2 + A_3 - A_4}{(x'_1 - ix'_4)^2} \epsilon^4 \\
&- 8 \frac{(A_1 + A_2 - A_3 - A_4)(x'_1 - ix'_4) - (A_1 - A_2 + 2A_3 - 2A_4)(x'_2 - ix'_3)}{(x'_1 - ix'_4)^3} \epsilon^5 \\
&+ O(\epsilon^6), \\
V_{[2,1,1]}^A &= \frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3(x_3 - ix_4)}{(x_3 + ix_4)^3} + \frac{A_4}{(x_3 + ix_4)^2} = \frac{(4A_1 - 4A_2 + A_4)}{(x'_1 - ix'_4)^2} \epsilon^4 \\
&- \frac{[(8A_1 + 8A_2 + A_3)(x'_1 - ix'_4) + 4(-2A_1 + 2A_2 - A_4)(x'_2 - ix'_3)]}{(x'_1 - ix'_4)^3} \epsilon^5 + O(\epsilon^6), \\
V_{[2,2]}^A &= \frac{A_1}{(x_1 + ix_2)^2} + \frac{A_2(x_1 - ix_2)}{(x_1 + ix_2)^3} + \frac{A_3}{(x_3 + ix_4)^2} + \frac{A_4(x_3 - ix_4)}{(x_3 + ix_4)^3} \\
&= \frac{A_1 + A_3}{(x'_1 - ix'_4)^2} \epsilon^4 + \left( \frac{(2A_1 + 4A_3)(x'_2 - ix'_3)}{(x'_1 - ix'_4)^3} + \frac{(A_2 - A_4)}{(x'_1 - ix'_4)^2} \right) \epsilon^5 + O(\epsilon^6).
\end{aligned}$$

### 3.6 [3,1] to [4]

This specific contraction is not needed because already the  $[1, 1, 1, 1] \rightarrow [4]$  contraction takes the system  $V[3, 1]$  to  $V[4]$ .

### 3.7 [2,1,1] to [4]

This specific contraction is not needed because already the  $[1, 1, 1, 1] \rightarrow [4]$  contraction takes the system  $V[2, 1, 1]$  to  $V[4]$ .

### 3.8 [1,1,1,1] to [3,1]

$$\begin{aligned}
-L'_{12} + iL'_{24} &= -a\sqrt{2a^2 - 2} \epsilon L_{12} \\
L'_{13} &= -\frac{i}{\sqrt{a^2 - 1}}(L_{13} + aL_{12}), \\
L'_{14} + iL'_{34} &= \sqrt{2} a \epsilon L_{14}, \\
-L'_{12} + iL'_{23} &= i\sqrt{2} a \epsilon L_{23}, \\
L'_{24} &= i(\sqrt{a^2 - 1} L_{24} - iaL_{14}), \\
-L'_{14} + iL'_{34} &= \frac{\sqrt{2}}{\epsilon a\sqrt{a^2 - 1}} \left( L_{34} - \sqrt{a^2 - 1} L_{14} - iaL_{24} \right).
\end{aligned} \tag{56}$$

**Coordinate implementation:**

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{2} a \epsilon} (x'_1 + i x'_3) + \frac{x'_2}{a} + \frac{a \epsilon}{\sqrt{2}} (x'_1 - i x'_3), \\ x_2 &= \frac{i(x'_1 + i x'_2)}{\sqrt{2a^2 - 2} \epsilon}, \\ x_3 &= -\frac{(x'_1 + i x'_3)}{\sqrt{2a^2 - 2} a \epsilon} + \frac{\sqrt{a^2 - 1}}{a} x'_2, \quad x_4 = x'_4, \end{aligned}$$

where  $a$  is a parameter such that  $a(a-1) \neq 0$ .

**Limit of 2D potential:**

$$V_{[1,1,1,1]} \xrightarrow{\epsilon \rightarrow 0} V_{[31]},$$

where  $V[31]$  is given by (45) and

$$\begin{aligned} a_1 &= \frac{c_1}{2\epsilon^2} + \frac{c_3}{4a^4\epsilon^4}, \quad a_2 = \frac{c_2}{4\sqrt{2}(a^2-1)^2\epsilon^3} + \frac{c_3}{4(a^2-1)^2\epsilon^4}, \\ a_3 &= \frac{c_2}{4\sqrt{2}(a^2-1)^2a^2\epsilon^3} + \frac{c_3}{4(a^2-1)^2a^4\epsilon^4}, \quad a_4 = c_4. \end{aligned}$$

**Basis of conformal symmetries for original system:**

$$H_0 + V_{[1,1,1,1]}, \quad Q_{12}, \quad Q_{13},$$

where

$$Q_{jk} = (x_j \partial_{x_k} - x_k \partial_{x_j})^2 + a_j \frac{x_k^2}{x_j^2} + a_k \frac{x_j^2}{x_k^2}, \quad 1 \leq j < k \leq 4.$$

**Contracted basis:**

$$\begin{aligned} H_0 + V_{[1,1,1,1]} &\rightarrow H'_0 + V_{[3,1]}, \\ \epsilon^2 \left( Q_{12} + \frac{c_3}{2a^2(a^2-1)\epsilon^4} + \frac{\sqrt{2}c_2}{a^2(a^2-1)\epsilon^3} \right) &\rightarrow -\frac{c_1}{2(a^2-1)} \\ -\frac{2c_3x_2'^2}{a^2(a^2-1)(x'_1 + i x'_3)^2} - \frac{c_2}{2a^2(a^2-1)(x'_1 + i x'_3)} - \frac{1}{2a^2(a^2-1)}(L'_{12} - i L'_{23})^2, \\ \epsilon \left( Q_{13} + a^2 Q_{12} + \frac{(a^2-1)c_3}{2a^4\epsilon^4} + \frac{\sqrt{2}c_2}{8a^2\epsilon^3} + \frac{c_1(a^2-1)}{2\epsilon^2} \right) &\rightarrow \frac{\sqrt{2}c_1x_2'}{x'_1 + i x'_3} \\ + \frac{\sqrt{2}c_2(4x_2'^2 + x_4'^2)}{4(x'_1 + i x'_3)^2} + \frac{2\sqrt{2}c_3x_2'(2x_2'^2 + x_4'^2)}{(x'_1 + i x'_3)^3} + \frac{i\sqrt{2}}{2}\{L'_{13}, L'_{12} - i L'_{23}\}. \end{aligned}$$

If we apply the  $[1,1,1,1] \rightarrow [3,1]$  contraction to the  $[2,1,1]$  system, the system contracts to one with potential  $V[3,1]$ , but with different parameters, or to  $[0]$ .

If we apply the same contraction to the  $[2,2]$  system, the system again contracts to one with potential  $V[0]$ , but with different parameters.

If we apply the same contraction to the  $[3,1]$  system, the system contracts to itself, but with different parameters.

If we apply the same contraction to the  $[4]$  system, the system contracts to the system with potential  $V[0]$ , (47), but with altered parameters.

If we apply the same contraction to the  $[0]$  system, the system contracts to the system with potential  $V[0]$ , (47), but with altered parameters.

If we apply this same contraction to the (1) system, (57) it becomes a potential conformally equivalent to (58).

If we apply this same contraction to the (2) system, (58) it contracts to itself with altered parameters.

### 3.8.1 [1,1,1,1] to [3,1] contraction and Stäckel transforms

For fixed  $A_j$  we have the expansions

$$\begin{aligned}
V_{[1,1,1,1]}^A &= \frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3}{x_3^2} + \frac{A_4}{x_4^2} = \\
&\frac{A_4}{x_4^2} + \frac{2(A_2 + (A_1 - A_2 - A_3)a^2 + A_3a^4)\epsilon^2}{(x_1 + ix_3)^2} \\
&+ \frac{4\sqrt{2}a^2x_2(A_3 - A_1 - 2A_3a^2 + A_3a^4)\epsilon^3}{(x_1 + ix_3)^3} \\
&- \frac{4a^2(A_1a^2x_1^2 + (-3A_1 + 3A_3(1 - a^2))x_2^2 + A_1a^2x_3^2)\epsilon^4}{(x_1 + ix_3)^4} + O(\epsilon^5), \\
V_{[2,1,1]}^A &= \frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3(x_3 - ix_4)}{(x_3 + ix_4)^3} + \frac{A_4}{(x_3 + ix_4)^2} \\
&= \frac{2(A_1a^2 + A_2(1 - a^2) + (A_3 + A_4)a^2(1 - a^2)^2)}{(x'_1 + ix'_3)^2}\epsilon^2 + O(\epsilon^3), \\
V_{[2,2]}^A &= \frac{A_1}{(x_1 + ix_2)^2} + \frac{A_2(x_1 - ix_2)}{(x_1 + ix_2)^3} + \frac{A_3}{(x_3 + ix_4)^2} + \frac{A_4(x_3 - ix_4)}{(x_3 + ix_4)^3} \\
&= \frac{k_1A_1 + k_2A_2 + k_3(A_3 + A_4)}{(x'_1 + ix'_3)^2}\epsilon^2 + O(\epsilon^3), \quad k_1, k_2, k_3 \text{ generic}, \\
V_{[3,1]}^A &= \frac{A_1}{(x_3 + ix_4)^2} + \frac{A_2x_1}{(x_3 + ix_4)^3} + \frac{A_3(4x_1^2 + x_2^2)}{(x_3 + ix_4)^4} + \frac{A_4}{x_2^2} \\
&= \left[ \frac{2A_1a^2 - 2A_2a^2\sqrt{a^2 - 1} + 4A_3a^2(3a^2 - 4) - 2A_4}{(x'_1 + ix'_3)^2} \right] (a^2 - 1)\epsilon^2 + O(\epsilon^4),
\end{aligned}$$

### 3.8.2 Conformal Stäckel transforms of the (1) system

We write potential  $V(1)$  in the form

$$V(1) = a_1 \frac{1}{(x_1 + ix_2)^2} + a_2 \frac{1}{(x_3 + ix_4)^2} + a_3 \frac{(x_3 + ix_4)}{(x_1 + ix_2)^3} + a_4 \frac{(x_3 + ix_4)^2}{(x_1 + ix_2)^4} \quad (57)$$

and designate it  $(a_1, a_2, a_3, a_4)$ , defining the conformally superintegrable system [1]. For every choice of  $(a_1, a_2, a_3, a_4)$  the potential  $V(1)$  generates a conformal Stäckel transform to a special case of  $E15$ , always flat.

### 3.8.3 Conformal Stäckel transforms of the (2) system

We write potential  $V(2)'$  in the normalized form

$$V(2)' = a_1 \frac{1}{(x_3 + ix_4)^2} + a_2 \frac{(x_1 + ix_2)}{(x_3 + ix_4)^3} + a_3 \frac{(x_1 + ix_2)^2}{(x_3 + ix_4)^4} + a_4 \frac{(x_1 + ix_2)^3}{(x_3 + ix_4)^5} \quad (58)$$

and designate it  $(a_1, a_2, a_3, a_4)$ , defining the conformally superintegrable system [2]. For every choice of  $(a_1, a_2, a_3, a_4)$  the potential  $V(2)'$  generates a conformal Stäckel transform to a special case of  $E15$ , always flat.

## 4 Helmholtz contractions from Bôcher contractions

We describe how Bôcher contractions of conformal superintegrable systems induce contractions of Helmholtz superintegrable systems. The basic idea here is that the procedure of taking a conformal Stäckel transform of a conformal superintegrable system, followed by a Helmholtz contraction yields the same result as taking a Bôcher contraction followed by an ordinary Stäckel transform: The diagrams commute. We illustrate with an example.

We consider the conformal Stäckel transforms of the conformal system  $[1, 1, 1, 1]$  with potential  $V_{[1,1,1,1]}$ . The various possibilities are listed in subsection 3.1.1. Let  $H$  be the initial Hamiltonian. In terms of tetraspherical coordinates the conformal Stäckel transformed potential will take the form

$$V = \frac{\frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2} + \frac{a_4}{x_4^2}}{\frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3}{x_3^2} + \frac{A_4}{x_4^2}} = \frac{V_{[1,1,1,1]}}{F(\mathbf{x}, \mathbf{A})},$$

where

$$F(\mathbf{x}, \mathbf{A}) = \frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3}{x_3^2} + \frac{A_4}{x_4^2},$$

and the transformed Hamiltonian will be

$$\hat{H} = \frac{1}{F(\mathbf{x}, \mathbf{A})} H,$$

where the transform is determined by the fixed vector  $(A_1, A_2, A_3, A_4)$ . Now we apply the Bôcher contraction  $[1, 1, 1, 1] \rightarrow [2, 1, 1]$  to this system. In the limit as  $\epsilon \rightarrow 0$  the potential  $V_{[1,1,1,1]} \rightarrow V_{[2,1,1]}$ , (39), and  $H \rightarrow H'$  the  $[2, 1, 1]$  system. Now consider

$$F(\mathbf{x}(\epsilon), \mathbf{A}) = V'(\mathbf{x}', A)\epsilon^\alpha + O(\epsilon^{\alpha+1}),$$

where the integer exponent  $\alpha$  depends upon our choice of  $\mathbf{A}$ . We will provide the theory to show that the system defined by Hamiltonian

$$\hat{H}' = \lim_{\epsilon \rightarrow 0} \epsilon^\alpha \hat{H}(\epsilon) = \frac{1}{V'(\mathbf{x}', A)} H'$$

is a superintegrable system that arises from the system  $[2, 1, 1]$  by a conformal Stäckel transform induced by the potential  $V'(\mathbf{x}', A)$ . Thus the Helmholtz superintegrable system with potential  $V = V_{[1,1,1,1]}/F$  contracts to the Helmholtz superintegrable system with potential  $V_{[2,1,1]}/V'$ . The contraction is induced by a generalized Inönü-Wigner Lie algebra contraction of the conformal algebra  $so(4, \mathbb{C})$ . In this case the possibilities for  $V'$  can be read off from the expression in subsection 3.1.2. Then the  $V'$  can be identified with a  $[2, 1, 1]$  potential from the list in subsection 3.1.3. The results follow. For each  $\mathbf{A}$  corresponding to a constant curvature or Darboux superintegrable system  $O$  we list the contracted system  $O'$  and  $\alpha$ . For Koenigs spaces we will not go into detail but merely give the contraction for a “generic” Koenigs system: One for which there are no rational numbers  $r_j$ , not all 0, such that  $\sum_{j=1}^4 r_j A_j = 0$ . This ensures that the contraction is also “generic”.

**Example 2** In Section 3.1.2, first equation, consider the Stäckel transform  $(1, 0, 0, 0)$ , i.e.,  $1/x_1^2$ . The transformed system is

$$H = \frac{1}{x_1^2} \left( \sum_{i=1}^4 \partial_{x_i}^2 \right) + \frac{1}{x_1^2} \left( \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2} + \frac{a_4}{x_4^2} \right)$$

which is S9. Now take the  $[1, 1, 1, 1] \rightarrow [2, 1, 1]$  Bocher contraction, equation (39). The sum of the derivatives in  $H$  goes to  $\sum_{i=1}^4 \partial_{x'_i}^2$  and the numerator of the potential goes to equation (39). However, the denominator  $1/x_1^2$  goes as

$$1/x_1^2 = -2\epsilon^2 / ((x'_1 + ix'_2)^2 + O(\epsilon^6))$$

from the first equation in Section 3.1.2, case  $A_1 = 1, A_2 = 0, A_3 = 0, A_4 = 0$ . Thus, if we set  $H' = \epsilon^2 H$  and go to the limit as  $\epsilon \rightarrow 0$ , we get a contracted system with potential  $b_1 + b_2(x^2 + y^2) + b_3/x^2 + b_4/y^2$  in Cartesian coordinates, up to a scalar factor  $-2$ . This is E1.

#### 4.1 Contraction $[1,1,1,1]$ to $[2,1,1]$ applied to conformal Stäckel transforms of system V $[1,1,1,1]$ .

1.

$$\begin{aligned} \mathbf{A} &= (1, 0, 0, 0), (0, 1, 0, 0), \quad O = S_9 \rightarrow O' = E_1, \quad \alpha = 2, \\ \mathbf{A} &= (0, 0, 1, 0), (0, 0, 0, 1) \quad O = S_9 \rightarrow O' = S_2, \quad \alpha = 0, \end{aligned}$$

2.

$$\begin{aligned} \mathbf{A} &= (1, 1, 1, 1), \quad O = S_8 \rightarrow O' = S_4, \quad \alpha = 0, \\ \mathbf{A} &= (0, 1, 0, 1), (1, 0, 1, 0) \quad O = S_8 \rightarrow O' = S_2, \quad \alpha = 0, \end{aligned}$$

3.

$$\begin{aligned} \mathbf{A} &= (0, 0, 1, 1), \quad O = S_7 \rightarrow O' = S_4, \quad \alpha = 0, \\ \mathbf{A} &= (1, 1, 0, 0), \quad O = S_7 \rightarrow O' = E_{16}, \quad \alpha = 4, \end{aligned}$$

4.

$$\begin{aligned} \mathbf{A} &= (A_1, A_2, 0, 0), (A_1 A_2 \neq 0, A_1 \neq A_2), \quad O = D4B \rightarrow O' = E_1, \quad \alpha = 2, \\ &= (0, 0, A_1, A_2), \quad O = D4B \rightarrow O' = D4A, \quad \alpha = 0, \\ \mathbf{A} &= \text{all other permutations}, \quad O = D4B \rightarrow O' = S_2, \quad \alpha = 0, \end{aligned}$$

5.

$$\begin{aligned} \mathbf{A} &= (1, 1, A, A), (A, A, 1, 1), \quad A \neq 0, \quad O = D4C \rightarrow O' = S_4, \quad \alpha = 0, \\ \mathbf{A} &= \text{all other permutations}, \quad O = D4C \rightarrow O' = D4A, \quad \alpha = 0, \end{aligned}$$

6.

$$\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[1, 1, 1, 1] \rightarrow O' = D4A, \quad \alpha = 0.$$

**Comment 5** *Already in this example we are able to characterize contractions of Darboux systems in a manner completely analogous to those of constant curvature systems. That wasn't possible before we extended our method to conformally superintegrable systems.*

## 4.2 Contraction [1,1,1,1] to [2,2] applied to conformal Stäckel transforms of system V[1,1,1,1].

The target systems are conformal Stäckel transforms of  $V_{[2,2]}$ . Partial results are:

1.

$$\mathbf{A} = (1, 0, 0, 0) \text{ and all permutations, } O = S_9 \rightarrow O' = E_7, \quad \alpha = 2,$$

2.

$$\mathbf{A} = (1, 1, 1, 1), (0, 1, 1, 0) \quad O = S_8 \rightarrow O' = E_{19}, \quad \alpha = 4,$$

$$\mathbf{A} = (0, 1, 0, 1), (1, 0, 1, 0) \quad O = S_8 \rightarrow O' = E_7, \quad \alpha = 2,$$

$$\mathbf{A} = (1, 0, 0, 1) \quad O = S_8 \rightarrow O' = E_{17}, \quad \alpha = 2,$$

3.

$$\mathbf{A} = (0, 0, 1, 1), \quad O = S_7 \rightarrow O' = E_{17}, \quad \alpha = 4,$$

$$\mathbf{A} = (1, 1, 0, 0), \quad O = S_7 \rightarrow O' = E_{19}, \quad \alpha = 4,$$

4.

$$\mathbf{A} = (0, 0, A_3, A_4), \quad A_3 A_4 \neq 0, A_3 \neq A_4, \text{ and all permutations,}$$

$$O = D4B \rightarrow O' = E_7, \quad \alpha = 2,$$

5.

$$\mathbf{A} = (1, 1, A, A), \quad A \neq 0, \text{ and all permutations,}$$

$$O = D4C \rightarrow O' = E_{19}, \quad \alpha = 1,$$

6.

$$\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[1, 1, 1, 1] \rightarrow O' = E_7, \quad \alpha = 2.$$

Additional results can be obtained for this contraction and the following by permuting the coordinate indices of the image potential before applying the contraction.

## 4.3 Contraction [1,1,1,1] to [3,1] applied to conformal Stäckel transforms of system V[1,1,1,1].

The target systems are conformal Stäckel transforms of  $V_{[3,1]}$ . Partial results are (assuming generic  $a$ ):

1.

$$\mathbf{A} = (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), \quad O = S_9 \rightarrow O' = E_2, \quad \alpha = 2,$$

$$\mathbf{A} = (0, 0, 0, 1), \quad O = S_9 \rightarrow O' = S_1, \quad \alpha = 0,$$

2.

$$\mathbf{A} = (1, 1, 1, 1), (0, 1, 0, 1), (1, 0, 0, 1) \quad O = S_8 \rightarrow O' = S_1, \quad \alpha = 0,$$

$$\mathbf{A} = (1, 0, 1, 0), (0, 1, 1, 0) \quad O = S_8 \rightarrow O' = E_2, \quad \alpha = 2,$$

3.

$$\mathbf{A} = (0, 0, 1, 1), \quad O = S_7 \rightarrow O' = S_1, \quad \alpha = 0,$$

$$\mathbf{A} = (1, 1, 0, 0), \quad O = S_7 \rightarrow O' = E_2, \quad \alpha = 2,$$

4.

$$\mathbf{A} = (0, 0, A_3, A_4), (A_3, 0, 0, A_4), (0, A_3, 0, A_4), \quad A_3 A_4 \neq 0, A_3 \neq A_4,$$

$$O = D4B \rightarrow O' = S_1, \quad \alpha = 0,$$

$$\mathbf{A} = (A_1, A_2, 0, 0), (A_1, 0, A_2, 0), (0, A_1, A_2, 0), \quad A_1 A_2 \neq 0, A_1 \neq A_2,$$

$$O = D4B \rightarrow O' = E_2, \quad \alpha = 2,$$

5.

$$\mathbf{A} = (1, 1, A, A), \text{ and all permutations, } A \neq 0, 1,$$

$$O = D4C \rightarrow O' = S_1, \quad \alpha = 0,$$

6.

$$\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[1, 1, 1, 1] \rightarrow O' = S_1, \quad \alpha = 0.$$

#### 4.4 Contraction [1,1,1,1] to [4] applied to conformal Stäckel transforms of system V[1,1,1,1].

The target systems are conformal Stäckel transforms of  $V_{[4]}$ . Partial results are (generic in the parameters  $a, b$ ):

1.

$$\mathbf{A} = (1, 0, 0, 0), \text{ and all permutations, } \quad O = S_9 \rightarrow O' = E_{10}, \quad \alpha = 6,$$

2.

$$\mathbf{A} = (1, 1, 1, 1), (0, 1, 0, 1), (1, 0, 0, 1), \quad O = S_8 \rightarrow O' = E_{10}, \quad \alpha = 6,$$

$$\mathbf{A} = (1, 0, 1, 0), (0, 1, 1, 0) \quad O = S_8 \rightarrow O' = E_{10}, \quad \alpha = 6,$$

3.

$$\mathbf{A} = (0, 0, 1, 1), (1, 1, 0, 0), \quad O = S_7 \rightarrow O' = E_{10}, \quad \alpha = 6,$$

4.

$$\mathbf{A} = (0, 0, A_3, A_4), \text{ and all permutations, } A_3 A_4 \neq 0, A_3 \neq A_4,$$

$$O = D4B \rightarrow O' = E_{10}, \quad \alpha = 6,$$

5.

$$\mathbf{A} = (1, 1, A, A), \text{ and all permutations, } A \neq 0, 1,$$

$$O = D4C \rightarrow O' = E_{10}, \quad \alpha = 6,$$

6.

$$\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[1, 1, 1, 1] \rightarrow O' = E_{10}, \quad \alpha = 6.$$



#### 4.5 Contraction [2,2] to [4] applied to conformal Stäckel transforms of system V[1,1,1,1].

The target systems are conformal Stäckel transforms of  $V_{[4]}$ . Partial results are:

1.

$$\mathbf{A} = (1, 0, 0, 0), \text{ and all permutations, } O = S_9 \rightarrow O' = E_{10}, \quad \alpha = 4,$$

2.

$$\begin{aligned} \mathbf{A} &= (1, 1, 1, 1), \quad O = S_8 \rightarrow O' = E_9, \quad \alpha = 6, \\ \mathbf{A} &= (0, 1, 0, 1), (1, 0, 1, 0), \quad O = S_8 \rightarrow O' = E_{10}, \quad \alpha = 4, \\ \mathbf{A} &= (0, 1, 1, 0), (1, 0, 0, 1), \quad O = S_8 \rightarrow O' = E_9, \quad \alpha = 5, \end{aligned}$$

3.

$$\mathbf{A} = (0, 0, 1, 1), (1, 1, 0, 0), \quad O = S_7 \rightarrow O' = E_{10}, \quad \alpha = 5,$$

4.

$$\begin{aligned} \mathbf{A} &= (0, 0, A_3, A_4), \text{ and all permutations, } A_3 A_4 \neq 0, A_3 \neq A_4, \\ O &= D4B \rightarrow O' = E_{10}, \quad \alpha = 4, \end{aligned}$$

5.

$$\begin{aligned} \mathbf{A} &= (1, 1, A, A), \text{ and all permutations } A \neq 0, 1, \\ O &= D4C \rightarrow O' = E_{10}, \quad \alpha = 5, \end{aligned}$$

6.

$$\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[1, 1, 1, 1] \rightarrow O' = E_{10}, \quad \alpha = 4.$$

Note that, although the values of  $\alpha$  differ, the target systems agree with those for  $[1, 1, 1, 1] \rightarrow [4]$  contractions of  $V_{[1,1,1,1]}$ , except in the single case  $S_8 \rightarrow E_9$ .

#### 4.6 Contraction [2,1,1] to [3,1] applied to conformal Stäckel transforms of system V[1,1,1,1].

The target systems are conformal Stäckel transforms of  $V_{[3,1]}$ . Partial results are:

1.

$$\begin{aligned} \mathbf{A} &= (1, 0, 0, 0), (0, 1, 0, 0), \quad O = S_9 \rightarrow O' = E_2, \quad \alpha = 6, \\ \mathbf{A} &= (0, 0, 1, 0), \quad O = S_9 \rightarrow O' = E_2, \quad \alpha = 4, \\ \mathbf{A} &= (0, 0, 0, 1), \quad O = S_9 \rightarrow O' = S_1, \quad \alpha = 0, \end{aligned}$$

2.

$$\begin{aligned} \mathbf{A} &= (1, 1, 1, 1), (0, 1, 0, 1), (1, 0, 0, 1) \quad O = S_8 \rightarrow O' = S_1, \quad \alpha = 0, \\ \mathbf{A} &= (1, 0, 1, 0), (0, 1, 1, 0) \quad O = S_8 \rightarrow O' = E_2, \quad \alpha = 4, \end{aligned}$$

3.

$$\begin{aligned} \mathbf{A} &= (0, 0, 1, 1), \quad O = S_7 \rightarrow O' = S_1, \quad \alpha = 0, \\ \mathbf{A} &= (1, 1, 0, 0), \quad O = S_7 \rightarrow O' = E_2, \quad \alpha = 8, \end{aligned}$$

4.

$$\mathbf{A} = (0, 0, A_3, A_4), (A_3, 0, 0, A_4), (0, A_3, 0, A_4), \quad A_3 A_4 \neq 0, A_3 \neq A_4,$$

$$O = D4B \rightarrow O' = S_1, \quad \alpha = 0,$$

$$\mathbf{A} = (A_1, A_2, 0, 0), (0, A_1, A_2, 0) \quad A_1 A_2 \neq 0, A_1 \neq A_2,$$

$$O = D4B \rightarrow O' = E_2, \quad \alpha = 6,$$

$$\mathbf{A} = (A_1, 0, A_3, 0), \quad A_1 A_3 \neq 0, A_1 \neq A_3,$$

$$O = D4B \rightarrow O' = E_2, \quad \alpha = 4,$$

5.

$$\mathbf{A} = (1, 1, A, A), \text{ and all permutations, } A \neq 0, 1,$$

$$O = D4C \rightarrow O' = S_1, \quad \alpha = 0,$$

6.

$$\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[1, 1, 1, 1] \rightarrow O' = S_1, \quad \alpha = 0.$$

Note that, although the values of  $\alpha$  differ, the target systems agree with those for  $[1, 1, 1, 1] \rightarrow [3, 1]$  contractions of  $V_{[1,1,1,1]}$ .

#### 4.7 Contraction $[1,1,1,1]$ to $[2,1,1]$ applied to conformal Stäckel transforms of system $V[2,1,1]$ .

The target systems are conformal Stäckel transforms of  $V_{[2,1,1]}$ . Partial results are:

1.

$$\mathbf{A} = (1, 1, 0, 0), \quad O = S_4 \rightarrow O' = S_4, \quad \alpha = 0,$$

2.

$$\mathbf{A} = (1, 0, 0, 0), (0, 1, 0, 0), \quad O = S_2 \rightarrow O' = S_2, \quad \alpha = 0,$$

3.

$$\mathbf{A} = (0, 0, 0, 1), \quad O = E_1 \rightarrow O' = E_1, \quad \alpha = 2,$$

4.

$$\mathbf{A} = (0, 0, 1, 0), \quad O = E_{16} \rightarrow O' = E_{16}, \quad \alpha = 4,$$

5.

$$\mathbf{A} = (A_3, A_4, 0, 0), (A_3 A_4 \neq 0, A_3 \neq A_4),$$

$$O = D4A \rightarrow O' = D4A, \quad \alpha = 0,$$

6.

$$\mathbf{A} = (0, 0, A_1, A_2), (A_1 A_2 \neq 0), \quad O = D3B \rightarrow O' = E_1, \quad \alpha = 2,$$

7.

$$\mathbf{A} = (A, 0, 0, 1), (0, A, 0, 1) \quad A \neq 0, \quad O = D2B \rightarrow O' = S_2, \quad \alpha = 0,$$

8.

$$\mathbf{A} = (1, 1, A, 0), \quad A \neq 0, \quad O = D2C \rightarrow O' = S_4, \quad \alpha = 0,$$

9.

$$\mathbf{A} = (A_3, A_4, A_2, A_1), \quad O = K[2, 1, 1] \rightarrow O' = S_4, \quad \alpha = 0.$$

#### 4.8 Contraction [1,1,1,1] to [2,2] applied to conformal Stäckel transforms of system V[2,1,1].

The target systems are conformal Stäckel transforms of  $V_{[2,2]}$ . Partial results are:

1.  $\mathbf{A} = (1, 1, 0, 0), \quad O = S_4 \rightarrow O' = E_{19}, \quad \alpha = 4,$
2.  $\mathbf{A} = (1, 0, 0, 0), (0, 1, 0, 0), \quad O = S_2 \rightarrow O' = E_7, \quad \alpha = 2,$
3.  $\mathbf{A} = (0, 0, 0, 1), \quad O = E_1 \rightarrow O' = E_8, \quad \alpha = 2,$
4.  $\mathbf{A} = (0, 0, 1, 0), \quad O = E_{16} \rightarrow O' = E_{17}, \quad \alpha = 4,$
5.  $\mathbf{A} = (A_1, A_2, 0, 0), (A_1 A_2 \neq 0, A_1 \neq A_2),$   
 $O = D4A \rightarrow O' = E_7, \quad \alpha = 2,$
6.  $\mathbf{A} = (0, 0, A_3, A_4), (A_3 A_4 \neq 0), \quad O = D3B \rightarrow O' = E_8, \quad \alpha = 2,$
7.  $\mathbf{A} = (A, 0, 0, 1), (0, A, 0, 1) \quad A \neq 0, \quad O = D2B \rightarrow O' = E_7, \quad \alpha = 2,$
8.  $\mathbf{A} = (1, 1, A, 0), \quad A \neq 0, \quad O = D2C \rightarrow O' = E_{19}, \quad \alpha = 4,$
9.  $\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[2, 1, 1] \rightarrow O' = E_7, \quad \alpha = 2,$

#### 4.9 Contraction [1,1,1,1] to [3,1] applied to conformal Stäckel transforms of system V[2,1,1].

The target systems are conformal Stäckel transforms of  $V_{[3,1]}$ . Generically in  $a$ , partial results are:

1.  $\mathbf{A} = (1, 1, 0, 0), \quad O = S_4 \rightarrow O' = E_2, \quad \alpha = 2,$
2.  $\mathbf{A} = (1, 0, 0, 0), (0, 1, 0, 0), \quad O = S_2 \rightarrow O' = E_2, \quad \alpha = 2,$
3.  $\mathbf{A} = (0, 0, 0, 1), \quad O = E_1 \rightarrow O' = E_2, \quad \alpha = 2,$
4.  $\mathbf{A} = (0, 0, 1, 0), \quad O = E_{16} \rightarrow O' = E_2, \quad \alpha = 2,$
5.  $\mathbf{A} = (A_1, A_2, 0, 0), (A_1 A_2 \neq 0, A_1 \neq A_2),$   
 $O = D4A \rightarrow O' = E_2, \quad \alpha = 2,$
6.  $\mathbf{A} = (0, 0, A_3, A_4), (A_3 A_4 \neq 0), \quad O = D3B \rightarrow O' = E_2, \quad \alpha = 2,$

7.

$$\mathbf{A} = (A, 0, 0, 1), (0, A, 0, 1) \quad A \neq 0, \quad O = D2B \rightarrow O' = E_2, \quad \alpha = 2,$$

8.

$$\mathbf{A} = (1, 1, A, 0), \quad A \neq 0, \quad O = D2C \rightarrow O' = E_2, \quad \alpha = 2,$$

9.

$$\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[2, 1, 1] \rightarrow O' = E_2, \quad \alpha = 2,$$

#### 4.10 Contraction [1,1,1,1] to [4] applied to conformal Stäckel transforms of system V[2,1,1].

The target systems are conformal Stäckel transforms of  $V_{[4]}$ . Partial results are:

Stäckel transforms of  $V_{[3,1]}$ . Generically in  $a$ , the results are:

1.

$$\mathbf{A} = (1, 1, 0, 0), \quad O = S_4 \rightarrow O' = E_{10}, \quad \alpha = 6,$$

2.

$$\mathbf{A} = (1, 0, 0, 0), (0, 1, 0, 0), \quad O = S_2 \rightarrow O' = E_{10}, \quad \alpha = 6,$$

3.

$$\mathbf{A} = (0, 0, 0, 1), \quad O = E_1 \rightarrow O' = E_{10}, \quad \alpha = 6,$$

4.

$$\mathbf{A} = (0, 0, 1, 0), \quad O = E_{16} \rightarrow O' = E_{10}, \quad \alpha = 6,$$

5.

$$\mathbf{A} = (A_1, A_2, 0, 0), (A_1 A_2 \neq 0, A_1 \neq A_2), \\ O = D4A \rightarrow O' = E_{10}, \quad \alpha = 6,$$

6.

$$\mathbf{A} = (0, 0, A_3, A_4), (A_3 A_4 \neq 0), \quad O = D3B \rightarrow O' = E_{10}, \quad \alpha = 6,$$

7.

$$\mathbf{A} = (A, 0, 0, 1), (0, A, 0, 1) \quad A \neq 0, \quad O = D2B \rightarrow O' = E_{10}, \quad \alpha = 6,$$

8.

$$\mathbf{A} = (1, 1, A, 0), \quad A \neq 0, \quad O = D2C \rightarrow O' = E_{10}, \quad \alpha = 6,$$

9.

$$\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[2, 1, 1] \rightarrow O' = E_{10}, \quad \alpha = 6,$$

#### 4.11 Contraction [2,2] to [4] applied to conformal Stäckel transforms of system V[2,1,1].

The target systems are conformal Stäckel transforms of  $V_{[4]}$ . Partial results are:

1.

$$\mathbf{A} = (1, 1, 0, 0), \quad O = S_4 \rightarrow O' = E_{10}, \quad \alpha = 5,$$

2.

$$\mathbf{A} = (1, 0, 0, 0), (0, 1, 0, 0), \quad O = S_2 \rightarrow O' = E_{10}, \quad \alpha = 4,$$

3.  $\mathbf{A} = (0, 0, 0, 1), \quad O = E_1 \rightarrow O' = E_{10}, \quad \alpha = 4,$
4.  $\mathbf{A} = (0, 0, 1, 0), \quad O = E_{16} \rightarrow O' = E_{10}, \quad \alpha = 5,$
5.  $\mathbf{A} = (A_1, A_2, 0, 0), (A_1 A_2 \neq 0, A_1 \neq A_2),$   
 $O = D4A \rightarrow O' = E_{10}, \quad \alpha = 4,$
6.  $\mathbf{A} = (0, 0, A_3, A_4), (A_3 A_4 \neq 0), \quad O = D3B \rightarrow O' = E_{10}, \quad \alpha = 4,$
7.  $\mathbf{A} = (A, 0, 0, 1), (0, A, 0, 1) \quad A \neq 0,$   
 $O = D2B \rightarrow O' = E_{10}, (\text{generically}) \quad \alpha = 4,$   
 $O = D2B \rightarrow O' = E_9, (\text{special case}) \quad \alpha = 5,$
8.  $\mathbf{A} = (1, 1, A, 0), \quad A \neq 0, \quad O = D2C \rightarrow O' = E_{10}, (\text{generically}) \quad \alpha = 5,$
9.  $\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[2, 1, 1] \rightarrow O' = E_{10}, \quad \alpha = 4,$

#### 4.12 Contraction [2,1,1] to [3,1] applied to conformal Stäckel transforms of system V[2,1,1].

The target systems are conformal Stäckel transforms of  $V_{[3,1]}$ . Partial results are:

1.  $\mathbf{A} = (1, 1, 0, 0), \quad O = S_4 \rightarrow O' = E_2, \quad \alpha = 8,$
2.  $\mathbf{A} = (1, 0, 0, 0), (0, 1, 0, 0), \quad O = S_2 \rightarrow O' = E_2, \quad \alpha = 6,$
3.  $\mathbf{A} = (0, 0, 0, 1), \quad O = E_1 \rightarrow O' = E_2, \quad \alpha = 4,$
4.  $\mathbf{A} = (0, 0, 1, 0), \quad O = E_{16} \rightarrow O' = E_2, \quad \alpha = 4,$
5.  $\mathbf{A} = (A_1, A_2, 0, 0), (A_1 A_2 \neq 0, A_1 \neq A_2),$   
 $O = D4A \rightarrow O' = E_2, \quad \alpha = 6,$
6.  $\mathbf{A} = (0, 0, A_3, A_4), (A_3 A_4 \neq 0),$   
 $O = D3B \rightarrow O' = E_2, (\text{generic}) \quad \alpha = 4,$
7.  $\mathbf{A} = (A, 0, 0, 1), (0, A, 0, 1) \quad A \neq 0,$   
 $O = D2B \rightarrow O' = E_2, (\text{generically}) \quad \alpha = 6,$
8.  $\mathbf{A} = (1, 1, A, 0), \quad A \neq 0, \quad O = D2C \rightarrow O' = E_2 \quad \alpha = 4,$
9.  $\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[2, 1, 1] \rightarrow O' = E_2, \quad \alpha = 4,$

#### 4.13 Contraction [1,1,1,1] to [2,1,1] applied to conformal Stäckel transforms of system V[2,2].

The target systems are conformal Stäckel transforms of  $V_{[2,2]}$ . Partial results are:

1.  $\mathbf{A} = (0, 0, 1, 0), \quad O = E_8 \rightarrow O' = E_8, \quad \alpha = 0, )$
2.  $\mathbf{A} = (0, 0, 0, 1), \quad O = E_{17} \rightarrow O' = E_{17}, \quad \alpha = 0,$
3.  $\mathbf{A} = (1, 0, A_3, 0), \quad O = E_7 \rightarrow O' = E_7, \quad \alpha = 0, (\text{generically}$
4.  $\mathbf{A} = (0, 1, 0, A_4), \quad O = E_{19} \rightarrow O' = E_{19}, \quad \alpha = 0, (\text{generically}$
5.  $\mathbf{A} = (0, 0, A_3, A_4, ), (A_3 A_4 \neq 0), \quad O = D3C \rightarrow O' = D3C, \quad \alpha = 0,$
6.  $\mathbf{A} = (A_1, A_2, 0, 0), (A_1 A_2 \neq 0), \quad O = D3D \rightarrow O' = E_7, \quad \alpha = 2,$
7.  $\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[2, 2] \rightarrow O' = D3C, \quad \alpha = 0,$

#### 4.14 Contraction [1,1,1,1] to [2,2] applied to conformal Stäckel transforms of system V[2,2].

The target systems are conformal Stäckel transforms of  $V_{[2,2]}$ . Partial results are:

1.  $\mathbf{A} = (0, 0, 1, 0), \quad O = E_8 \rightarrow O' = E_8, \quad \alpha = 2, )$
2.  $\mathbf{A} = (0, 0, 0, 1), \quad O = E_{17} \rightarrow O' = E_{17}, \quad \alpha = 2,$
3.  $\mathbf{A} = (1, 0, A_3, 0), \quad O = E_7 \rightarrow O' = E_7, \quad \alpha = 2, (\text{generically}$
4.  $\mathbf{A} = (0, 1, 0, A_4), \quad O = E_{19} \rightarrow O' = E_{19}, \quad \alpha = 4, (\text{generically}$
5.  $\mathbf{A} = (0, 0, A_3, A_4, ), (A_3 A_4 \neq 0), \quad O = D3C \rightarrow O' = E_8, \quad \alpha = 2,$
6.  $\mathbf{A} = (A_1, A_2, 0, 0), (A_1 A_2 \neq 0), \quad O = D3D \rightarrow O' = E_7, \quad \alpha = 2,$
7.  $\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[2, 2] \rightarrow O' = E_7, \quad \alpha = 2,$

#### 4.15 Contraction [1,1,1,1] to [3,1] applied to conformal Stäckel transforms of system V[2,2].

The target systems are conformal Stäckel transforms of  $V_{[3,1]}$ . Partial results are:

1.  $\mathbf{A} = (0, 0, 1, 0), \quad O = E_8 \rightarrow O' = E'_3, \quad \alpha = 2,$
2.  $\mathbf{A} = (0, 0, 0, 1), \quad O = E_{17} \rightarrow O' = E'_3, \quad \alpha = 2,$
3.  $\mathbf{A} = (1, 0, A_3, 0), \quad O = E_7 \rightarrow O' = E'_3, \quad \alpha = 2, \text{ (generically)}$
4.  $\mathbf{A} = (0, 1, 0, A_4), \quad O = E_{19} \rightarrow O' = E'_3, \quad \alpha = 2, \text{ (generically)}$
5.  $\mathbf{A} = (0, 0, A_3, A_4), (A_3 A_4 \neq 0),$   
 $O = D3C \rightarrow O' = E'_3, \quad \alpha = 2, \text{ (generically)}$   
 $O = D3C \rightarrow O' = D1C, \quad \alpha = 3, \text{ (special case)}$
6.  $\mathbf{A} = (A_1, A_2, 0, 0), (A_1 A_2 \neq 0),$   
 $O = D3D \rightarrow O' = E'_3, \quad \alpha = 2, \text{ (generically)}$
7.  $\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[2, 2] \rightarrow O' = E'_3, \quad \alpha = 2,$

#### 4.16 Contraction [1,1,1,1] to [4] applied to conformal Stäckel transforms of system V[2,2].

Partial results:

1.  $\mathbf{A} = (0, 0, 1, 0), \quad O = E_8 \rightarrow O' = E'_3, \quad \alpha = 6,$
2.  $\mathbf{A} = (0, 0, 0, 1), \quad O = E_{17} \rightarrow O' = E'_3, \quad \alpha = 6,$
3.  $\mathbf{A} = (1, 0, A_3, 0), \quad O = E_7 \rightarrow O' = E'_3, \quad \alpha = 6,$
4.  $\mathbf{A} = (0, 1, 0, A_4), \quad O = E_{19} \rightarrow O' = E'_3, \quad \alpha = 6,$
5.  $\mathbf{A} = (0, 0, A_3, A_4), (A_3 A_4 \neq 0), \quad O = D3C \rightarrow O' = E'_3, \quad \alpha = 6,$
6.  $\mathbf{A} = (A_1, A_2, 0, 0), (A_1 A_2 \neq 0), \quad O = D3D \rightarrow O' = E'_3, \quad \alpha = 6,$
7.  $\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[2, 2] \rightarrow O' = E'_3, \quad \alpha = 6,$

#### 4.17 Contraction [2,2] to [4] applied to conformal Stäckel transforms of system V[2,2].

The target systems are conformal Stäckel transforms of  $V_{[4]}$ . Partial results:

1.  $\mathbf{A} = (0, 0, 1, 0), \quad O = E_8 \rightarrow O' = E_{10}, \quad \alpha = 4,$
2.  $\mathbf{A} = (0, 0, 0, 1), \quad O = E_{17} \rightarrow O' = E_{10}, \quad \alpha = 5,$
3.  $\mathbf{A} = (1, 0, A_3, 0), \quad O = E_7 \rightarrow O' = E_{10}, \quad \alpha = 4, \text{ (generically)}$   
 $O = E_7 \rightarrow O' = E_9, \quad \alpha = 5, \text{ (special case)}$
4.  $\mathbf{A} = (0, 1, 0, A_4), \quad O = E_{19} \rightarrow O' = E_{10}, \quad \alpha = 5, \text{ (generically)}$   
 $O = E_{19} \rightarrow O' = E_9, \quad \alpha = 6, \text{ (special case)}$
5.  $\mathbf{A} = (0, 0, A_3, A_4, ), (A_3 A_4 \neq 0), \quad O = D3C \rightarrow O' = E_{10}, \quad \alpha = 4,$
6.  $\mathbf{A} = (A_1, A_2, 0, 0), (A_1 A_2 \neq 0), \quad O = D3D \rightarrow O' = E_{10}, \quad \alpha = 4,$
7.  $\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[2, 2] \rightarrow O' = E_{10}, \quad \alpha = 4,$

#### 4.18 Contraction [2,1,1] to [3,1] applied to conformal Stäckel transforms of system V[2,2].

The target systems are conformal Stäckel transforms of  $V_{[0]}$ . Partial results:

1.  $\mathbf{A} = (0, 0, 1, 0), \quad O = E_8 \rightarrow O' = E'_3, \quad \alpha = 4,$
2.  $\mathbf{A} = (0, 0, 0, 1), \quad O = E_{17} \rightarrow O' = E'_3, \quad \alpha = 4,$
3.  $\mathbf{A} = (1, 0, A_3, 0), \quad O = E_7 \rightarrow O' = E'_3, \quad \alpha = 2,$
4.  $\mathbf{A} = (0, 1, 0, A_4), \quad O = E_{19} \rightarrow O' = E'_3, \quad \alpha = 0,$
5.  $\mathbf{A} = (0, 0, A_3, A_4, ), (A_3 A_4 \neq 0),$   
 $O = D3C \rightarrow O' = E'_3, \quad \alpha = 4, \text{ (generically)}$   
 $O = D3C \rightarrow O' = D1C, \quad \alpha = 6, \text{ (special case)}$
6.  $\mathbf{A} = (A_1, A_2, 0, 0), (A_1 A_2 \neq 0), \quad O = D3D \rightarrow O' = E'_3, \quad \alpha = 0,$
7.  $\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[2, 2] \rightarrow O' = E'_3, \quad \alpha = 0,$



#### 4.19 Contraction $[1,1,1,1]$ to $[2,1,1]$ applied to conformal Stäckel transforms of system $V[3,1]$ .

The target systems are conformal Stäckel transforms of the singular system  $V(1)$ . All systems are flat space and Stäckel equivalent to special cases of  $E15$ .

#### 4.20 Contraction $[1,1,1,1]$ to $[2,2]$ applied to conformal Stäckel transforms of system $V[3,1]$ .

The target systems are conformal Stäckel transforms of the singular system  $V(1)$ . All systems are flat space and Stäckel equivalent to special cases of  $E15$ .

#### 4.21 Contraction $[1,1,1,1]$ to $[3,1]$ applied to conformal Stäckel transforms of system $V[3,1]$ .

The target systems are conformal Stäckel transforms of  $V_{[3,1]}$ . Partial results are:

1.  $\mathbf{A} = (0, 0, 0, 1), \quad O = S_1 \rightarrow O' = E_2, \quad \alpha = 2,$
2.  $\mathbf{A} = (1, 0, 0, 0), \quad O = E_2 \rightarrow O' = E_2, \quad \alpha = 2,$
3.  $\mathbf{A} = (a, 1, 0, 0), \quad O = D1B \rightarrow O' = E_2, \quad \alpha = 2,$
4.  $\mathbf{A} = (0, 0, 1, 0), \quad O = D2A \rightarrow O' = E_2, \text{ (generically) } \alpha = 2,$
5.  $\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[3, 1] \rightarrow O' = E_2, \quad \alpha = 2.$

#### 4.22 Contraction $[1,1,1,1]$ to $[4]$ applied to conformal Stäckel transforms of system $V[3,1]$ .

The target systems are conformal Stäckel transforms of  $V_{[4]}$ . Partial results are:

1.  $\mathbf{A} = (0, 0, 0, 1), \quad O = S_1 \rightarrow O' = E_{10}, \quad \alpha = 6,$
2.  $\mathbf{A} = (1, 0, 0, 0), \quad O = E_2 \rightarrow O' = E_{10}, \quad \alpha = 6,$
3.  $\mathbf{A} = (a, 1, 0, 0), \quad O = D1B \rightarrow O' = E_{10}, \quad \alpha = 6,$
4.  $\mathbf{A} = (0, 0, 1, 0), \quad O = D2A \rightarrow O' = E_{10}, \quad \alpha = 6,$
5.  $\mathbf{A} = (A_1, A_2, A_3, A_4), \quad O = K[3, 1] \rightarrow O' = E_{10}, \quad \alpha = 6.$

#### 4.23 Contraction [2,2] to [4] applied to conformal Stäckel transforms of system V[3,1].

The target systems are conformal Stäckel transforms of the singular system  $V_{[2]}$ . All systems are flat space and Stäckel equivalent to special cases of  $E_{15}$ .

#### 4.24 Contraction [2,1,1] to [3,1] applied to conformal Stäckel transforms of system V[3,1].

The target systems are conformal Stäckel transforms of  $V_{[3,1]}$ . Partial results:

1.  $\mathbf{B} = (0, 0, 0, 1), \quad O = S_1 \rightarrow O' = S_1, \quad \alpha = 0,$
2.  $\mathbf{B} = (1, 0, 0, 0), \quad O = E_2 \rightarrow O' = E_2, \quad \alpha = 6,$
3.  $\mathbf{B} = (a, 1, 0, 0), \quad O = D1B \rightarrow O' = E_2, \quad \alpha = 6,$
4.  $\mathbf{B} = (0, 0, 1, 0), \quad O = D2A \rightarrow O' = E_2, \quad \alpha = 6,$
5.  $\mathbf{B} = (B_1, B_2, B_3, B_4), \quad O = K[3, 1] \rightarrow O' = S_1, \quad \alpha = 0.$

#### 4.25 Contraction [1,1,1,1] to [2,1,1] applied to conformal Stäckel transforms of system V[4].

The target systems are conformal Stäckel transforms of  $V_{[0]}$ . Partial results are:

1.  $\mathbf{D} = (1, D_2, 0, 0), \quad O = E_{10} \rightarrow O' = E'_3, \quad \alpha = 2, )$
2.  $\mathbf{D} = (0, 1, 0, 0), \quad O = E_9 \rightarrow O' = E_{11}, \quad \alpha = 3,$
3.  $\mathbf{D} = (0, 0, 0, 1), \quad O = D1A \rightarrow O' = E_{20}, \quad \alpha = 4,$
4.  $\mathbf{D} = (D_1, D_2, D_3, D_4), \quad O = K[4] \rightarrow O' = E'_3, \quad \alpha = 2,$

#### 4.26 Contraction [1,1,1,1] to [2,2] applied to conformal Stäckel transforms of system V[4].

The target systems are conformal Stäckel transforms of the singular system  $V(2)$ . All systems are flat space and Stäckel equivalent to special cases of  $E_{15}$ .

#### 4.27 Contraction [1,1,1,1] to [3,1] applied to conformal Stäckel transforms of system V[4].

The target systems are conformal Stäckel transforms of  $V_{[0]}$ . Partial results are:

1.  $\mathbf{C} = (1, C_2, 0, 0), \quad O = E_{10} \rightarrow O' = E'_3, \quad \alpha = 2,$
2.  $\mathbf{C} = (0, 1, 0, 0), \quad O = E_9 \rightarrow O' = E'_3, \quad \alpha = 2,$
3.  $\mathbf{C} = (0, 0, 0, 1), \quad O = D1A \rightarrow O' = E'_3, \quad \alpha = 2,$
4.  $\mathbf{C} = (C_1, C_2, C_3, C_4), \quad O = K[4] \rightarrow O' = E'_3, \quad \alpha = 2,$

#### 4.28 Contraction [1,1,1,1] to [4] applied to conformal Stäckel transforms of system V[4].

The target systems are conformal Stäckel transforms of  $V_{[0]}$ . Partial results are:

1.  $\mathbf{C} = (1, C_2, 0, 0), \quad O = E_{10} \rightarrow O' = E'_3, \quad \alpha = 6,$
2.  $\mathbf{C} = (0, 1, 0, 0), \quad O = E_9 \rightarrow O' = E'_3, \quad \alpha = 6,$
3.  $\mathbf{C} = (0, 0, 0, 1), \quad O = D1A \rightarrow O' = E'_3, \quad \alpha = 6,$
4.  $\mathbf{C} = (C_1, C_2, C_3, C_4), \quad O = K[4] \rightarrow O' = E'_3, \quad \alpha = 6,$

#### 4.29 Contraction [2,2] to [4] applied to conformal Stäckel transforms of system V[4].

The target systems are conformal Stäckel transforms of the singular system  $V_{(2)}$ . All systems are flat space and Stäckel equivalent to special cases of  $E_{15}$ .

#### 4.30 Contraction [2,1,1] to [3,1] applied to conformal Stäckel transforms of system V[4].

The target systems are conformal Stäckel transforms of  $V_{[0]}$ . Partial results:

1.  $\mathbf{C} = (1, C_2, 0, 0), \quad O = E_{10} \rightarrow O' = E'_3, \quad \alpha = 1, (a \neq 0), 0, (a = 0)$
2.  $\mathbf{C} = (0, 1, 0, 0), \quad O = E_9 \rightarrow O' = E'_3, \quad \alpha = 1,$
3.  $\mathbf{C} = (0, 0, 0, 1), \quad O = D1A \rightarrow O' = E'_3, \quad \alpha = -1,$
4.  $\mathbf{C} = (C_1, C_2, C_3, C_4), \quad O = K[4] \rightarrow O' = E3', \quad \alpha = -1,$

### 4.31 Contraction [1,1,1,1] to [2,1,1] applied to conformal Stäckel transforms of system V[0].

The target systems are conformal Stäckel transforms of  $V_{[0]}$ . Partial results are:

1.

$$\begin{aligned}\mathbf{C} &= ((C_2^2 + C_3^2)/4, C_2, C_3, 1), \quad O = E_{20} \rightarrow O' = E'_3, \\ &\quad (C_2^2 + C_3^2 \neq 0) \quad \alpha = 2, \\ O &= E_{20} \rightarrow O' = E_{11}, (C_2^2 + C_3^2 = 0, C_2 C_3 \neq 0) \quad \alpha = 3, \\ O &= E_{20} \rightarrow O' = D3A, (C_2 = C_3 = 0) \quad \alpha = 4,\end{aligned}$$

2.

$$\begin{aligned}\mathbf{C} &= (C_1, 1, \pm i, 0), \quad O = E_{11} \rightarrow O' = E'_3, (C_1 \neq 0) \quad \alpha = 2, \\ O &= E_{11} \rightarrow O' = E_{11}, (C_1 = 0) \quad \alpha = 3,\end{aligned}$$

3.

$$\mathbf{C} = (1, 0, 0, 0), \quad O = E'_3 \rightarrow O' = E'_3, \quad \alpha = 2$$

4.

$$\begin{aligned}\mathbf{C} &= (C_1, C_2, C_3, 0), \quad (C_2^2 + C_3^2 \neq 0), \\ O &= D1C \rightarrow O' = E'_3, (C_1 \neq 0) \quad \alpha = 2, \\ O &= D1C \rightarrow O' = D1C, (C_1 = 0) \quad \alpha = 3,\end{aligned}$$

5.

$$\begin{aligned}\mathbf{C} &= (C_1, C_2, C_3, 1), \quad (4C_1 \neq C_2^2 + C_3^2), \\ O &= D3A \rightarrow O' = E'_3, (C_1 \neq 0) \quad \alpha = 2, \\ O &= D3A \rightarrow O' = D1C, (C_1 = 0) \quad \alpha = 3,\end{aligned}$$

6.

$$\mathbf{C} = (C_1, C_2, C_3, C_4), \quad O = K[0] \rightarrow O' = E'_3, \quad \alpha = 2,$$

### 4.32 Contraction [1,1,1,1] to [2,2] applied to conformal Stäckel transforms of system V[0].

The target systems are conformal Stäckel transforms of  $V_{[0]}$ . Partial results are:

1.

$$\begin{aligned}\mathbf{C} &= ((C_2^2 + C_3^2)/4, C_2, C_3, 1), \quad O = E_{20} \rightarrow O' = E'_3, \\ &\quad (C_2^2 + C_3^2 \neq 0) \quad \alpha = 2, \\ O &= E_{20} \rightarrow O' = E_{20}, (C_3 = -iC_2 \neq 0) \quad \alpha = 4, \\ O &= E_{20} \rightarrow O' = E'_3, (C_3 = iC_2 \neq 0) \quad \alpha = 2,\end{aligned}$$

2.

$$\begin{aligned}\mathbf{C} &= (C_1, 1, \pm i, 0), \quad O = E_{11} \rightarrow O' = E'_3, (C_1 \neq 0, C_3 = -i) \quad \alpha = 2, \\ O &= E_{11} \rightarrow O' = E_{11}, (C_3 = i) \quad \alpha = 2, \\ O &= E_{11} \rightarrow O' = E_{11}, (C_1 = 0, C_3 = -i) \quad \alpha = 4,\end{aligned}$$

3.

$$\mathbf{C} = (1, 0, 0, 0), \quad O = E'_3 \rightarrow O' = E'_3, \quad \alpha = 2$$

4.

$$\mathbf{C} = (C_1, C_2, C_3, 0), (C_2^2 + C_3^2 \neq 0), \quad O = D1C \rightarrow O' = D1C, \quad \alpha = 2,$$

5.

$$\begin{aligned} \mathbf{C} &= (C_1, C_2, C_3, 1), (4C_1 \neq C_2^2 + C_3^2), \quad O = D3A \rightarrow O' = E_{11}, \\ &\quad (C_1 \neq 0, C_2^2 + C_3^2 = 0) \quad \alpha = 2, \\ &\quad O = D3A \rightarrow O' = D1C, (C_2^2 + C_3^2 \neq 0) \quad \alpha = 2, \end{aligned}$$

6.

$$\mathbf{C} = (C_1, C_2, C_3, C_4), \quad O = K[0] \rightarrow O' = D1C, \quad \alpha = 2,$$

### 4.33 Contraction [1,1,1,1] to [3,1] applied to conformal Stäckel transforms of system V[0].

The target systems are conformal Stäckel transforms of  $V_{[0]}$ . Partial results are:

1.

$$\mathbf{C} = ((C_2^2 + C_3^2)/4, C_2, C_3, 1), \quad O = E_{20} \rightarrow O' = E'_3, \quad \alpha = 2,$$

2.

$$\begin{aligned} \mathbf{C} &= (C_1, 1, \pm i, 0), \quad O = E_{11} \rightarrow O' = E'_3, (\text{generic}) \quad \alpha = 2, \\ &\quad O = E_{11} \rightarrow O' = D1C, (\text{spacial case}) \quad \alpha = 3, \end{aligned}$$

3.

$$\mathbf{C} = (1, 0, 0, 0), \quad O = E'_3 \rightarrow O' = E'_3, \quad \alpha = 2$$

4.

$$\begin{aligned} \mathbf{C} &= (C_1, C_2, C_3, 0), (C_2^2 + C_3^2 \neq 0), \\ &\quad O = D1C \rightarrow O' = E'_3, (\text{generic}) \quad \alpha = 2, \\ &\quad O = D1C \rightarrow O' = D1C, (\text{special case}) \quad \alpha = 3, \end{aligned}$$

5.

$$\begin{aligned} \mathbf{C} &= (C_1, C_2, C_3, 1), (4C_1 \neq C_2^2 + C_3^2), \\ &\quad O = D3A \rightarrow O' = E'_3, (\text{generic}) \quad \alpha = 2, \end{aligned}$$

6.

$$\mathbf{C} = (C_1, C_2, C_3, C_4), \quad O = K[0] \rightarrow O' = E'_3, \quad \alpha = 2,$$

### 4.34 Contraction [1,1,1,1] to [4] applied to conformal Stäckel transforms of system V[0].

The target systems are conformal Stäckel transforms of  $V_{[0]}$ . Partial results are:

1.

$$\mathbf{C} = ((C_2^2 + C_3^2)/4, C_2, C_3, 1), \quad O = E_{20} \rightarrow O' = E'_3, \quad \alpha = 6,$$

2.

$$\mathbf{C} = (C_1, 1, \pm i, 0), \quad O = E_{11} \rightarrow O' = E'_3, \quad \alpha = 6,$$

3.

$$\mathbf{C} = (1, 0, 0, 0), \quad O = E'_3 \rightarrow O' = E'_3, \quad \alpha = 6$$

4.

$$\mathbf{C} = (C_1, C_2, C_3, 0), \quad (C_2^2 + C_3^2 \neq 0), \quad O = D1C \rightarrow O' = E'_3, \quad \alpha = 6,$$

5.

$$\mathbf{C} = (C_1, C_2, C_3, 1), \quad (4C_1 \neq C_2^2 + C_3^2), \quad O = D3A \rightarrow O' = E'_3, \alpha = 6,$$

6.

$$\mathbf{C} = (C_1, C_2, C_3, C_4), \quad O = K[0] \rightarrow O' = E'_3, \quad \alpha = 6,$$

#### 4.35 Contraction [2,2] to [4] applied to conformal Stäckel transforms of system V[0].

The target systems are conformal Stäckel transforms of  $V_{[0]}$ . Partial results are:

1.

$$\mathbf{C} = ((C_2^2 + C_3^2)/4, C_2, C_3, 1), \quad O = E_{20} \rightarrow O' = E'_3, \quad \alpha = 4,$$

2.

$$\mathbf{C} = (C_1, 1, \pm i, 0), \quad O = E_{11} \rightarrow O' = E'_3, \text{ (generic) } \alpha = 4,$$

$$O = E_{11} \rightarrow O' = E'_3, \text{ (specialcase) } \alpha = 5,$$

3.

$$\mathbf{C} = (1, 0, 0, 0), \quad O = E'_3 \rightarrow O' = E'_3, \quad \alpha = 4$$

4.

$$\mathbf{C} = (C_1, C_2, C_3, 0), \quad (C_2^2 + C_3^2 \neq 0), \\ O = D1C \rightarrow O' = E'_3, \text{ (generic) } \alpha = 4,$$

5.

$$\mathbf{C} = (C_1, C_2, C_3, 1), \quad (4C_1 \neq C_2^2 + C_3^2), \\ O = D3A \rightarrow O' = E'_3, \text{ (generic) } \alpha = 4,$$

6.

$$\mathbf{C} = (C_1, C_2, C_3, C_4), \quad O = K[0] \rightarrow O' = E'_3, \quad \alpha = 4,$$

#### 4.36 Contraction [2,1,1] to [3,1] applied to conformal Stäckel transforms of system V[0].

The target systems are conformal Stäckel transforms of  $V_{[0]}$ . Partial results:

1.

$$\mathbf{C} = ((C_2^2 + C_3^2)/4, C_2, C_3, 1), \quad O = E_{20} \rightarrow O' = E'_3 \quad \alpha = 6,$$

2.

$$\mathbf{C} = (C_1, 1, \pm i, 0), \quad O = E_{11} \rightarrow O' = E'_3, \text{ (generic) } \alpha = 6,$$

3.  $\mathbf{C} = (1, 0, 0, 0), \quad O = E'_3 \rightarrow O' = E'_3, \quad \alpha = 6$
4.  $\mathbf{C} = (C_1, C_2, C_3, 0), \quad (C_2^2 + C_3^2 \neq 0),$   
 $O = D1C \rightarrow O' = E'_3, \text{ (generic)} \quad \alpha = 6,$
5.  $\mathbf{C} = (C_1, C_2, C_3, 1), \quad (4C_1 \neq C_2^2 + C_3^2),$   
 $O = D3A \rightarrow O' = E'_3, \text{ (generic)} \quad \alpha = 6,$
6.  $\mathbf{C} = (C_1, C_2, C_3, C_4), \quad O = K[0] \rightarrow O' = E'_3, \quad \alpha = 6,$

## 5 Summary of the 8 Laplace superintegrable systems with nondegenerate potentials

All systems are of the form  $\left(\sum_{j=1}^4 \partial_{x_j}^2 + V(\mathbf{x})\right) \Psi = 0$ , or  $\left(\partial_x^2 + \partial_y^2 + \tilde{V}\right) \Psi = 0$  as a flat space system in Cartesian coordinates. The potentials are:

$$V_{[1,1,1,1]} = \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2} + \frac{a_4}{x_4^2}, \quad (59)$$

$$\begin{aligned} \tilde{V}_{[1,1,1,1]} &= \frac{a_1}{x^2} + \frac{a_2}{y^2} + \frac{4a_3}{(x^2 + y^2 - 1)^2} - \frac{4a_4}{(x^2 + y^2 + 1)^2}, \\ V_{[2,1,1]} &= \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3(x_3 - ix_4)}{(x_3 + ix_4)^3} + \frac{a_4}{(x_3 + ix_4)^2}, \\ \tilde{V}_{[2,1,1]} &= \frac{a_1}{x^2} + \frac{a_2}{y^2} - a_3(x^2 + y^2) + a_4, \end{aligned} \quad (60)$$

$$V_{[2,2]} = \frac{a_1}{(x_1 + ix_2)^2} + \frac{a_2(x_1 - ix_2)}{(x_1 + ix_2)^3} + \frac{a_3}{(x_3 + ix_4)^2} + \frac{a_4(x_3 - ix_4)}{(x_3 + ix_4)^3}, \quad (61)$$

$$\tilde{V}_{[2,2]} = \frac{a_1}{(x + iy)^2} + \frac{a_2(x - iy)}{(x + iy)^3} + a_3 - a_4(x^2 + y^2),$$

$$V_{[3,1]} = \frac{a_1}{(x_3 + ix_4)^2} + \frac{a_2x_1}{(x_3 + ix_4)^3} + \frac{a_3(4x_1^2 + x_2^2)}{(x_3 + ix_4)^4} + \frac{a_4}{x_2^2}, \quad (62)$$

$$\tilde{V}_{[3,1]} = a_1 - a_2x + a_3(4x^2 + y^2) + \frac{a_4}{y^2},$$

$$V_{[4]} = \frac{a_1}{(x_3 + ix_4)^2} + a_2 \frac{x_1 + ix_2}{(x_3 + ix_4)^3} + a_3 \frac{3(x_1 + ix_2)^2 - 2(x_3 + ix_4)(x_1 - ix_2)}{(x_3 + ix_4)^4} \quad (63)$$

$$+ a_4 \frac{4(x_3 + ix_4)(x_3^2 + x_4^2) + 2(x_1 + ix_2)^3}{(x_3 + ix_4)^5},$$

$$\tilde{V}_{[4]} = a_1 - a_2(x + iy) + a_3(3(x + iy)^2 + 2(x - iy)) - a_4(4(x^2 + y^2) + 2(x + iy)^3),$$

$$V_{[0]} = \frac{a_1}{(x_3 + ix_4)^2} + \frac{a_2x_1 + a_3x_2}{(x_3 + ix_4)^3} + a_4 \frac{x_1^2 + x_2^2}{(x_3 + ix_4)^4}, \quad (64)$$

$$\tilde{V}_{[0]} = a_1 - (a_2x + a_3y) + a_4(x^2 + y^2),$$

$$V_{arb} = \frac{1}{(x_3 + ix_4)^2} f\left(\frac{-x_1 - ix_2}{x_3 + ix_4}\right), \quad (65)$$

$$\tilde{V}_{arb} = f(x + iy), \text{ } f \text{ arbitrary}$$

$$V(1) = a_1 \frac{1}{(x_1 + ix_2)^2} + a_2 \frac{1}{(x_3 + ix_4)^2} + a_3 \frac{(x_3 + ix_4)}{(x_1 + ix_2)^3} + a_4 \frac{(x_3 + ix_4)^2}{(x_1 + ix_2)^4}, \quad (66)$$

$$\tilde{V}(1) = \frac{a_1}{(x + iy)^2} + a_2 - \frac{a_3}{(x + iy)^3} + \frac{a_4}{(x + iy)^4},$$

This is a special case of (65).

$$V(2)' = a_1 \frac{1}{(x_3 + ix_4)^2} + a_2 \frac{(x_1 + ix_2)}{(x_3 + ix_4)^3} + a_3 \frac{(x_1 + ix_2)^2}{(x_3 + ix_4)^4} + a_4 \frac{(x_1 + ix_2)^3}{(x_3 + ix_4)^5}, \quad (67)$$

$$\tilde{V}(2)' = a_1 + a_2(x + iy) + a_3(x + iy)^2 + a_4(x + iy)^3.$$

This is a special case of (65).

## 6 Summary of Stäckel equivalence classes of Helmholtz superintegrable systems

1. [1, 1, 1, 1]:

$$S9, S8, S7, D4B, D4C, K[1, 1, 1, 1]$$

2. [2, 1, 1]:

$$S4, S2, E1, E16, D4A, D3B, D2B, D2C, K[2, 1, 1]$$

3. [2, 2]:

$$E8, E17, E7, E19, D3C, D3D, K[2, 2]$$

4. [3, 1]:

$$S1, E2, D1B, D2A, K[3, 1]$$

5. [4]:

$$E10, E9, D1A, K[4]$$

6. [0]:

$$E20, E11, E3', D1C, D3A, K[0]$$

7. (1):

special cases of  $E15$

8. (2):

special cases of  $E15$



## 6.1 Summary of Bôcher contractions of Laplace systems

This is a summary of the results of applying each of the Bôcher contractions to each of the Laplace conformally superintegrable systems.

1.  $[1, 1, 1, 1] \rightarrow [2, 1, 1]$  contraction:

$$V_{[1,1,1,1]} \downarrow V_{[2,1,1]}; V_{[2,1,1]} \downarrow V_{[2,1,1]}, V_{[2,2]}, V_{[3,1]}; V_{[2,2]} \downarrow V_{[2,2]}, V_{[0]}; V_{[3,1]} \downarrow V_{(1)}, V_{[3,1]};$$

$$V_{[4]} \downarrow V_{[0]}, V_{(2)}; V_{[0]} \downarrow V_{[0]}; V_{(1)} \downarrow V_{(1)}, V_{(2)}; V_{(2)} \downarrow V_{(2)}.$$

2.  $[1, 1, 1, 1] \rightarrow [2, 2]$  contraction:

$$V_{[1,1,1,1]} \downarrow V_{[2,2]}; V_{[2,1,1]} \downarrow V_{[2,2]}, \text{ special case of } E_{15}; V_{[2,2]} \downarrow V_{[2,2]}, V_{[0]}; V_{[3,1]} \downarrow V_{(1)}, \text{ special case of } E_{15};$$

$$V_{[4]} \downarrow V_{(2)}; V_{[0]} \downarrow V_{[0]}; V_{(1)} \downarrow V_{(1)}, \text{ special case of } E_{15}; V_{(2)} \downarrow V_{(2)}.$$

3.  $[2, 1, 1] \rightarrow [3, 1]$  contraction:

$$V_{[1,1,1,1]} \downarrow V_{[3,1]}; V_{[2,1,1]} \downarrow V_{[3,1]}, V_{[0]}; V_{[2,2]} \downarrow V_{[0]}, V_{[3,1]} \downarrow V_{[3,1]}, V_{[0]}; V_{[4]} \downarrow V_{[0]};$$

$$V_{[0]} \downarrow V_{[0]}; V_{(1)} \downarrow V_{(2)}; V_{(2)} \downarrow V_{(2)}.$$

4.  $[1, 1, 1, 1] \rightarrow [4]$  contraction:

$$V_{[1,1,1,1]} \downarrow V_{[4]}; V_{[2,1,1]} \downarrow V_{[4]}; V_{[2,2]} \downarrow V_{[0]}; V_{[3,1]} \downarrow V_{[4]}; V_{[4]} \downarrow V_{[0]}, V_{[4]}; V_{[0]} \downarrow V_{[0]};$$

$$V_{(1)} \downarrow V_{(2)}; V_{(2)} \downarrow V_{(2)};$$

5.  $[2, 2] \rightarrow [4]$  contraction:

$$V_{[1,1,1,1]} \downarrow V_{[4]}; V_{[2,1,1]} \downarrow V_{[4]}, V_{(2)}; V_{[2,2]} \downarrow V_{[4]}, V_{[0]}; V_{[3,1]} \downarrow V_{(2)}; V_{[4]} \downarrow V_{(2)};$$

$$V_{[0]} \downarrow V_{[0]}, V_{(2)}; V_{(1)} \downarrow V_{(2)}; V_{(2)} \downarrow V_{(2)};$$

6.  $[1, 1, 1, 1] \rightarrow [3, 1]$  contraction:

$$V_{[1,1,1,1]} \downarrow V_{[3,1]}, V_{[2,1,1]} \downarrow V_{[3,1]}, V_{[0]}; V_{[2,2]} \downarrow V_{[0]}; V_{[3,1]} \downarrow V_{[3,1]}, V_{[0]}; V_{[4]} \downarrow V_{[0]}, V_{[0]} \downarrow V_{[0]},$$

$$V_{(1)} \downarrow V_{(2)}, V_{(2)} \downarrow V_{(2)}.$$

## 7 Summary of Helmholtz contractions

The superscript for each targeted Helmholtz system is the value of  $\alpha$ . In each table, corresponding to a single Laplace equation equivalence class, the top line is a list of the Helmholtz systems in the class, and the lower lines are the target systems under the Bôcher contraction. Contractions

of systems:

	[1,1,1,1]		equivalence	class	contractions	
contraction	$S_9$	$S_7$		$S_8$	$D_4 B$	$D_4 C$ $K[1111]$
[1111] $\downarrow$ [211]	$E_1^2$	$S_4^0$		$S_4^0$	$E_1^2$	$S_4^0$
	$S_2^0$	$S_2^0$		$E_{16}^0$	$D_4 A^0$	$D_4 A^0$
[1111] $\downarrow$ [22]	$E_7^2$	$E_{19}^4$		$E_{17}^4$	$E_7^2$	$E_{19}^1$
		$E_7^2$		$E_{19}^4$		$E_7^2$
		$E_{17}^2$				
[1111] $\downarrow$ [31]	$E_2^2$	$S_1^0$		$S_1^0$	$S_1^0$	$S_1^0$
	$S_1^0$	$E_2^2$		$E_2^2$	$E_2^2$	
[1111] $\downarrow$ [4]	$E_{10}^6$	$E_{10}^6$		$E_{10}^6$	$E_{10}^6$	$E_{10}^6$
[22] $\downarrow$ [4]	$E_{10}^4$	$E_9^6$		$E_{10}^5$	$E_{10}^4$	$E_{10}^5$
		$E_{10}^4$		—		$E_{10}^4$
		$E_9^5$				
[211] $\downarrow$ [31]	$E_2^6$	$S_1^0$		$S_1^0$	$S_1^0$	$S_1^0$
	$E_2^4$	$E_2^4$		$E_2^8$	$E_2^6$	
	$S_1^0$			$E_2^4$		

(68)

	[2,1,1]	equivalence	class	contractions					
contraction	$S_4$	$S_2$	$E_1$	$E_{16}$	$D_4A$	$D_3B$	$D_2B$	$D_2C$	$K[211]$
[1111] $\downarrow$ [211]	$S_4^0$ $E_{17}^4$ $S_1^0$	$S_2^0$ $E_8^2$ $S_1^0$ $E_2^2$	$E_1^2$ $E_8^0$ $E_2^2$	$E_{16}^4$ $E_{17}^0$ $E_2^2$	$D_4A^0$ $E_8^2$ $S_1^0$	$E_1^2$ $D_3C^0$ $E_2^2$ $D_1B^3$	$S_2^0$ $E_8^0$ $S_1^0$ $E_2^2$	$S_4^0$ $E_{17}^0$ $S_1^0$	$S_4^0$ $D_3C^0$ $S_1^0$
[1111] $\downarrow$ [22]	$E_{17}^4$	$E_8^2$	$E_8^2$	$E_{17}^4$	$E_7^2$ $E_8^2$	$E_8^2$ $E_{17}^2$	$E_7^2$ $E_8^2$	$E_{19}^4$ $E_{17}^4$	$E_7^2$
[1111] $\downarrow$ [31]	$S_1^0$ $E_3'^2$	$S_1^0$ $E_3'^2$	$E_2^2$ $E_3'^2$	$E_2^2$ $E_3'^2$	$S_1^0$ $E_3'^2$ $D_1C^3$	$E_2^2$ $D_1B^3$ $E_3'^2$ $D_1C^3$	$E_1^2$ $E_3'^2$ $D_1C^3$	$S_1^0$ $E_3'^2$	$S_1^0$ $E_3'^2$
[1111] $\downarrow$ [4]	$E_{10}^6$	$E_{10}^6$	$E_{10}^6$	$E_{10}^6$	$E_{10}^6$ $E_9^8$	$E_{10}^6$ $E_9^8$	$E_{10}^6$ $E_9^8$	$E_{10}^6$ $E_9^8$	$E_{10}^6$
[22] $\downarrow$ [4]	$E_{10}^5$	$E_{10}^4$	$E_{10}^4$	$E_{10}^5$	$E_{10}^4$	$E_{10}^4$ $E_{10}^5$	$E_{10}^4$ $E_{10}^5$	$E_{10}^4$	$E_{10}^4$
	Stäckel	transforms	of	$V(2)$					
[211] $\downarrow$ [31]	$S_1^0$ $E_3'^8$	$S_1^0$ $E_2^5$ $E_3'^6$	$E_2^6$ $E_3'^4$	$E_2^8$ $E_3'^4$	$S_1^0$ $E_3'^6$	$E_2^6$ $E_3'^6$	$S_1^0$ $E_2^5$ $E_3'^4$	$S_1^0$ $E_3'^4$	$S_1^0$ $E_3'^4$

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	[2,2]		equivalence	class	contractions			
contraction	$E_8$	$E_{17}$		$E_7$	$E_{19}$	$D_3C$	$D_3D$	$K[22]$
$[1111] \downarrow [211]$	$E_8^0$ $E_3'^2$	$E_{17}^0$ $E_3'^2$		$E_7^0$ $E_3'^2$	$E_{19}^0$ $E_3'^2$	$D_3C^0$ $E_3'^2$	$E_7^2$ $E_3'^2$	$D_3C^0$ $E_3'^2$
$[1111] \downarrow [22]$	$E_8^2$ $E_3'^2$	$E_{17}^4$ $E_{11}^2$		$E_7^2$ $E_3'^2$	$E_{19}^4$ $E_{11}^2$	$E_8^2$ $E_{11}^2$	$E_8^2$ $E_{11}^2$	$E_7^2$ $E_{11}^2$
$[1111] \downarrow [31]$	$E_3'^2$	$E_3'^2$		$E_3'^2$	$E_3'^2$ $E_{11}^4, E_{20}^4$	$E_3'^2$ $D_1C^3$	$E_3'^2$ $D_1C^3$	$E_3'^2$
$[1111] \downarrow [4]$	$E_3'^6$	$E_3'^6$		$E_3'^6$ $E_{11}^8$	$E_3'^6$ $E_{11}^8$	$E_3'^6$ $E_{11}^8$	$E_3'^6$ $E_{11}^8$	$E_3'^6$
$[22] \downarrow [4]$	$E_{10}^4$ $E_3'^2$	$E_{10}^5$ $E_{11}^1$		$E_{10}^4$ $E_9^5$ $E_3'^2$ $E_{11}^3$	$E_{10}^5$ $E_9^6$ $E_{11}^1$ $E_{20}^4$	$E_{10}^4$ $E_{11}^1$	$E_{10}^4$ $E_{11}^1$	$E_{10}^4$ $E_{11}^1$
$[211] \downarrow [31]$	$E_3'^4$ $E_3'^6$	$E_3'^4$ $E_3'^6$		$E_3'^2$ $E_3'^6$	$E_3'^2$ $E_{20}^4$	$E_3'^4$ $E_3'^6$ $D_1C^9$	$D_1C^2$ $E_3'^6$	$D_1C^2$ $E_3'^6$

(70)

	[3,1]		equivalence	class	contractions	
contraction	$S_1$		$E_2$	$D_1B$	$D_2A$	$K[31]$
$[1111] \downarrow [211]$	$S_1^0$	Stäckel	transforms $E_2^2$	of $E_2^2$ $D_1B^3$	$V(1)$ $E_2^2$ $D_2A^4$	$S_1^0$
$[1111] \downarrow [22]$		Stäckel	transforms	of	$V(1)$	
$[1111] \downarrow [31]$	$S_1^0$ $E_3'^2$		$E_2^2$ $E_3'^2$	$E_2^2$ $D_1B^3$ $E_3'^2$ $D_1C^3$	$E_2^2$ $E_3'^2$	$S_1^0$ $E_3'^2$
$[1111] \downarrow [4]$	$E_{10}^6$		$E_{10}^6$	$E_{10}^6$ $E_9^8$	$E_{10}^6$	$E_{10}^6$
$[22] \downarrow [4]$		Stäckel	transforms	of	$V(2)$	
$[211] \downarrow [31]$	$S_1^0$ $E_3'^4$		$E_2^6$ $E_2^2$ $E_3'^6$	$E_2^6$ $S_1^1$ $E_3'^6$	$E_2^6$ $S_1^0$ $E_3'^6$	$S_1^0$ $E_3'^4$

(71)

	[4]	equivalence	class	contractions
contraction	$E_{10}$	$E_9$	$D_1 A$	$K[4]$
$[1111] \downarrow [211]$	$E_3'^2$	$E_{11}^2$ $E_3'^2$	$E_{20}^4$ $E_3'^2$	$E_3'^2$
		Stäckel	transforms	of $V(2)$
$[1111] \downarrow [22]$	$E_3'^2$	Stäckel	transforms $E_3'^2$	of $V(2)$ $D_1 C^2$ $D_3 A^2$
$[1111] \downarrow [31]$	$E_3'^2$ $E_{11}^2$	$E_3'^2$	$E_3'^2$	$E_3'^2$
$[1111] \downarrow [4]$	$E_3'^6$ $E_{11}^8$ $E_{10}^6$ $E_9^8$	$E_3'^6$ $E_{10}^6$	$E_3'^6$ $E_{10}^6$	$E_3'^6$ $E_{10}^6$
$[22] \downarrow [4]$		Stäckel	transforms	of $V(2)$
$[211] \downarrow [31]$	$E_3'^1$ $E_3'^4$ $E_3'^6$	$E_3'^1$ $E_3'^5$ $E_3'^6$	$E_3'^{-1}$ $E_3'^4$ $E_3'^6$	$E_3'^{-1}$ $E_3'^3$ $E_3'^6$

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	[0]	equivalence	class	contractions		
contraction	$E_{20}$	$E_{11}$	$E_3'$	$D_1C$	$D_3A$	$K[0]$
$[1111] \downarrow [211]$	$E_3'^2$ $E_{11}^3$	$E_3'^2$ $E_{11}^3$	$E_3'^2$	$E_3'^2$ $D_1C^3$	$E_3'^2$ $D_1C^3$	$E_3'^2$
$[1111] \downarrow [22]$	$E_{11}^2$	$E_{11}^2$	$E_3'^2$	$E_{11}^2$	$E_{11}^2$ $E_3'^2$	$E_{11}^2$ $E_3'^2$
$[1111] \downarrow [31]$	$E_3'^2$	$E_3'^2$	$E_3'^2$	$E_3'^2$ $D_1C^3$	$E_3'^2$ $D_1C^3$	$E_3'^2$
$[1111] \downarrow [4]$	$E_3'^6$ $E_{11}^8$	$E_3'^6$ $E_{11}^8$	$E_3'^6$	$E_3'^6$ $E_{11}^8$	$E_3'^6$ $E_{11}^8$	$E_3'^6$
$[22] \downarrow [4]$	$E_3'^4$ $E_{11}^5$	$E_3'^4$ $E_{11}^5$	$E_3'^4$	$E_3'^4$ $E_{11}^5$	$E_3'^4$	$E_3'^4$
$[211] \downarrow [31]$	$E_3'^6$	$E_3'^6$	$E_3'^6$	$E_3'^6$ $D_1C^9$	$E_3'^6$	$E_3'^6$
(73)						

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